The goal of this text is to **explore how ASP programs with probabilistic facts** can lead to characterizations of the **joint distributions** of the program's atoms.

Notation

- The **complement** of x is $\overline{x} = 1 - x$.
- A **probabilistic atomic choice** $\alpha : a$ defines the disjunction $a \vee \neg a$ and assigns probabilities $P(a) = \alpha, P(\neg a) = \overline{\alpha}$.
- δa denotes the disjunction $a \vee \neg a$ associated to the probabilistic choice $\alpha : a$ and $\delta\{\alpha : a, a \in A\} = \{\delta a, a \in A\}$ for any set of atoms A.
- Start with the **closed world assumption**, where $\sim x \models \neg x$.
- Also assume that **probabilistic choices** and **subgoals** are iid.

- Let $\mathcal{A}$ be a set of **atoms**, $\mathcal{Z}$ the respective set of **observations**, $\mathcal{Z} = \left\{ z = \alpha \cup \nu \middle| \alpha \subseteq \mathcal{A} \wedge \nu \subseteq \{\neg a | a \in \mathcal{A}\} \right\}$ and $\mathcal{I}$ the set of consistent observations or **interpretations**, $\mathcal{I} = \left\{ z \in \mathcal{Z} \middle| \forall a \in \mathcal{A} \ \left| \{a, \neg a\} \cap z \right| \leq 1 \right\}$.
- A **PASP program** is $P = C \wedge F \wedge R$ and the sets of atoms, observations and interpretations of program P are denoted $\mathcal{A}_P, \mathcal{Z}_P$ and $\mathcal{I}_P$.
- $C = C_P = \{\alpha_i : a_i | i = 1 : n\}$ is a set of probabilistic atomic choices and δC_P the set of associated disjunctions, $F = F_P$ is a set of (common) facts and $R = R_P$ is a set of (common) rules.
- The **stable models** of $P = C \wedge F \wedge R$ are the stable models of $\delta P = \delta C + F + R$ and denoted $\mathcal{S} = \mathcal{S}_P$.

- **Proposition.** Let $x \in \mathcal{I}$ be an interpretation and $\langle x| = \{s \in \mathcal{S} | s \subseteq x\}$ and $|x\rangle = \{s \in \mathcal{S} | x \subseteq s\}$. Exactly one of the following cases takes place
  1. $\langle x| = \{x\} = |x\rangle$. If $a \in \langle x|$ and $b \in |x\rangle$ then $a \subseteq b$. Since stable models are minimal must be $a = b = x$ and x is a stable model.
  2. $\langle x| \neq \emptyset \wedge |x\rangle = \emptyset$.
  3. $\langle x| = \emptyset \wedge |x\rangle \neq \emptyset$.
  4. $\langle x| = \emptyset = |x\rangle$.

- The probabilistic facts C define a set $\Theta = \Theta_C$ of **total choices**, with 2^n elements, each one a set $\theta = \{c_1, \ldots, c_n\}$ where c_i is either a_i or $\neg a_i$.
- For each stable model $s \in \mathcal{S}$ let θ_s be the unique **total choice** contained in s and $\mathcal{S}_\theta \subseteq \mathcal{S}$ the set of stable models that contains θ.
- Define

$$p(\theta) = \prod_{a_i \in \theta} \alpha_i \prod_{\neg a_i \in \theta} \overline{\alpha_i}. \tag{1}$$

The problem we address is how to ***assign probabilities to observations*** *given that a total choice might entail zero or many stable models i.e. How to assign probabilities to the stable models of* $\mathcal{S}_\theta$ *when* $|\mathcal{S}_\theta| \neq 1$*?*

As it turns out, it is quite easy to come out with a program from which result no single probability distribution. For example

$$0.3 : a,$$
$$b \vee c \leftarrow a.$$

has three stable models

$$s_1 = \{\neg a\}$$
$$s_2 = \{a, b\}$$
$$s_3 = \{a, c\}$$

and while $p\big(\{\neg a\}\big) = 0.7$ is quite natural, we have no further information to support the choice of a singular $\alpha \in [0, 1]$ in the assignment

$$p\big(\{a, b\}\big) = 0.3\alpha$$
$$p\big(\{a, c\}\big) = 0.3\overline{\alpha}$$

Next we try to formalize the possible configurations of this scenario. Consider the ASP program $P = C \wedge F \wedge R$ with total choices Θ and stable models $\mathcal{S}$. Let $d : \mathcal{S} \rightarrow [0, 1]$ such that $\sum_{s \in \mathcal{S}_\theta} d(s) = 1$.

1. For each $z \in \mathcal{Z}$ only one of the following cases takes place
   1.1 z is inconsistent. Then **define**
   $$w_d(x) = 0. \tag{2}$$
   1.2 z is an interpretation and $\langle z| = \{z\} = |x\rangle$. Then $z = s$ is a stable model and **define**
   $$w_d(z) = w(s) = d(s)\, p(\theta_s)\,. \tag{3}$$
   1.3 z is an interpretation and $\langle z| \neq \emptyset \land |x\rangle = \emptyset$. Then **define**
   $$w_d(z) = \sum_{s \in \langle z|} w_d(s)\,. \tag{4}$$
   1.4 z is an interpretation and $\langle z| = \emptyset \land |z\rangle \neq \emptyset$. Then **define**
   $$w_d(z) = \prod_{s \in |z\rangle} w_d(s)\,. \tag{5}$$
   1.5 z is an interpretation and $\langle z| = \emptyset \land |z\rangle = \emptyset$. Then **define**
   $$w_d(z) = 0. \tag{6}$$
2. The last point defines a "weight" function on the observations that depends not only on the total choices and stable models of a PASP but also on a certain function d that must respect some conditions. To simplify the notation we use the

Consider the program:

$$c_1 = a \vee \neg a,$$
$$c_2 = b \vee c \leftarrow a.$$

This program has two total choices,

$$\theta_1 = \{\neg a\},$$
$$\theta_2 = \{a\}.$$

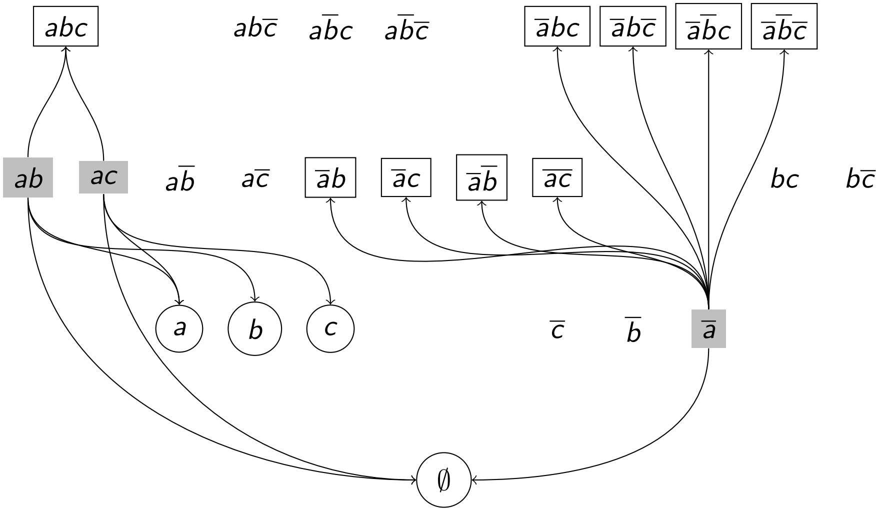
and three stable models,

$$s_1 = \{\neg a\},$$
$$s_2 = \{a, b\},$$
$$s_3 = \{a, c\}.$$

Suppose that we add an annotation $\alpha : a$, which entails $\overline{\alpha} : \neg a$. This is enough to get $w(s_1) = \overline{\alpha}$ but, on the absence of further information, no fixed probability can be assigned to either model s_2, s_3 except that the respective sum must be α. So, expressing our lack of knowledge using a parameter $d \in [0, 1]$ we get:

$$\begin{cases} w(s_1) = & \overline{\alpha} \\ w(s_2) = & d\alpha \\ w(s_3) = & \overline{d}\alpha. \end{cases}$$

Now consider all the interpretations for this program:



In this diagram:

- Negations are represented as *e.g.* $\overline{a}$ instead of $\neg a$; Stable models are denoted by shaded nodes as ab.
- Interpretations in $\langle x|$ are *e.g.* a and those in $|x\rangle$ are *e.g.* $\overline{a}b$. The remaining are simply denoted by *e.g.* $a\overline{b}$.
- The edges connect stable models with related interpretations. Up arrow indicate links to $|s\rangle$ and down arrows to $\langle s|$.
- The *weight propagation* sets:

$$
\begin{aligned}
w(abc) &= w(ab)\, w(ac) = \alpha^2 d\overline{d},\\
w(\overline{a}\cdot\cdot) &= w(\neg a) = \overline{\alpha},\\
w(a) &= w(ab) + w(ac) = \alpha(d + \overline{d}) = \alpha,\\
w(b) &= w(ab) = d\alpha,\\
w(c) &= w(ac) = \overline{d}\alpha,\\
w(\emptyset) &= w(ab) + w(ac) + w(\neg a) = d\alpha + \overline{d}\alpha + \overline{\alpha} = 1,\\
w\left(a\overline{b}\right) &= 0.
\end{aligned}
$$

- The total weight is

The following LP is non-stratified, because has a cycle with negated arcs:

$$c_1 = a \vee \neg a,$$
$$c_2 = b \leftarrow \sim c \wedge \sim a,$$
$$c_3 = c \leftarrow \sim b.$$

This program has three stable models

$$s_1 = \{a, c\},$$
$$s_2 = \{\neg a, b\},$$
$$s_3 = \{\neg a, c\}.$$

The disjunctive clause $a \vee \neg a$ defines a set of **total choices**

$$\Theta = \left\{\theta_1 = \{a\}, \theta_2 = \{\neg a\}\right\}.$$

Looking into probabilistic interpretations of the program and/or its models, we define $\alpha = P(\Theta = \theta_1) \in [0, 1]$ and $P(\Theta = \theta_2) = \overline{\alpha}$. Since s_1 is the only stable model that results from $\Theta = \theta_1$, it is natural to extend $P(s_1) = P(\Theta = \theta_1) = \alpha$. However, there is no clear way to assign $P(s_2), P(s_3)$ since *both models result from the single total choice* $\Theta = \theta_2$. Clearly,

$$P\left(s_2 \mid \Theta\right) + P\left(s_3 \mid \Theta\right) = \begin{cases} 0 & \text{if } \Theta = \theta_1 \\ 1 & \text{if } \Theta = \theta_2 \end{cases}$$

but further assumptions are not supported *a priori*. So let's **parameterize** the equation above,

$$\begin{cases} P\left(s_2 \mid \Theta = \theta_2\right) = & \beta \in [0, 1] \\ P\left(s_3 \mid \Theta = \theta_2\right) = & \overline{\beta}, \end{cases}$$

in order to explicit our knowledge, or lack of, with numeric values and relations.

Now we are able to define the **joint distribution** of the boolean random variables $A, B, C,$:

A, B, C	P	Obs.
$a, \neg b, c$	α	$s_1, \Theta = \theta_1$
$\neg a, b, \neg c$	$\overline{\alpha}\beta$	$s_2, \Theta = \theta_2$
$\neg a, \neg b, c$	$\overline{\alpha}\overline{\beta}$	$s_3, \Theta = \theta_2$
$*$	0	not stable models

where $\alpha, \beta \in [0, 1]$.

- We can use the basics of probability theory and logic programming to assign explicit *parameterized* probabilities to the (stable) models of a program.
- In the covered cases it was possible to define a (parameterized) *family of joint distributions*.
- How far this approach can cover all the cases on logic programs is (still) an issue *under investigation*.
- However, it is non-restrictive since *no unusual assumptions are made*.

- An **atom** is $r(t_1, \dots t_n)$ where
  - r is a n-ary predicate symbol and each t_i is a constant or a variable.
  - A **ground atom** has no variables; A **literal** is either an atom a or a negated atom $\neg a$.
- An **ASP Program** is a set of **rules** such as $h_1 \vee \cdots \vee h_m \leftarrow b_1 \wedge \cdots \wedge b_n$.
  - The **head** of this rule is $h_1 \vee \cdots \vee h_m$, the **body** is $b_1 \wedge \cdots \wedge b_n$ and each b_i is a **subgoal**.
  - Each h_i is a literal, each subgoal b_j is a literal or a literal preceded by $\sim$ and $m + n > 0$.
  - A **propositional program** has no variables.
  - A **non-disjunctive rule** has $m \leq 1$; A **normal rule** has $m = 1$; A **constraint** has $m = 0$; A **fact** is a normal rule with $n = 0$.
- The **Herbrand base** of a program is the set of ground literals that result from combining all the predicates and constants of the program.
  - An **interpretation** is a consistent subset (*i.e.* doesn't contain $\{a, \neg a\}$) of the Herbrand base.
  - Given an interpretation I, a ground literal a is **true**, $I \models a$, if $a \in I$; otherwise the literal is **false**.
  - A ground subgoal, $\sim b$, where b is a ground literal, is **true**, $I \models \sim b$, if $b \notin I$; otherwise, if $b \in I$, it is **false**.