A First-order Logic for Reasoning about Knowledge and Probability

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We present a first-order probabilistic epistemic logic, which allows combining operators of knowledge and probability within a group of possibly infinitely many agents. We define its syntax and semantics and prove the strong completeness property of the corresponding axiomatic system.[1]

CCS Concepts: • **Computing methodologies** → **Reasoning about belief and knowledge**; • **Theory of computation** → **Modal and temporal logics**; • **Computing methodologies** → **Probabilistic reasoning**;

Additional Key Words and Phrases: Probabilistic epistemic logic, strong completeness, coordinated actions, probabilistic common knowledge, infinite number of agents

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1 INTRODUCTION

Reasoning about knowledge is widely used in many applied fields such as computer science, artificial intelligence, economics, game theory, and so on [3, 4, 20, 27]. A particular line of research concerns the formalization in terms of multi-agent epistemic logics, which speak about knowledge about facts, but also about knowledge of other agents. One of the central notions is that of common knowledge, which has been shown as crucial for a variety of applications dealing with reaching agreements or coordinated actions [23]. Intuitively, φ is common knowledge of a group of agents exactly when everyone knows that everyone knows that everyone knows ... that φ is true.

However, it has been shown that in many practical systems common knowledge cannot be attained [19, 23]. This motivated some researchers to consider a weaker variant that still may be

[1]This article is a revised and extended version of the conference paper [47] presented at the 13th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'15), in which we introduced the propositional variant of the logic presented here, using a similar axiomatization technique.

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sufficient for carrying out a number of coordinated actions [27, 36, 42]. One of the approaches proposes a probabilistic variant of common knowledge [35], which assumes that coordinated actions hold with high probability. Possible applications of probabilistic common knowledge are discussed in the context of practical distributed systems and cryptographic protocols in References [25, 27, 32]. A propositional logical system that formalizes that notion is presented in Reference [17], where Fagin and Halpern developed a joint framework for reasoning about knowledge and probability and proposed a complete axiomatization.

We use Reference [17] as a starting point and generalize it in two ways:

First, we extend the propositional formalization from Reference [17] by allowing reasoning about knowledge and probability of events expressible in *a first-order language*. We use the most general approach, allowing arbitrary combination of standard epistemic operators, probability operators, first-order quantifiers, and, in addition, of probabilistic common knowledge operator. The need for first-order extension is recognized by epistemic and probability logic communities. Wolter [49] pointed out that first-order common knowledge logics are of interest both from the point of view of applications and of pure logic. He argued that first-order base is necessary whenever the application domains are infinite (like in epistemic analysis of the playability of games with mixed strategies) or finite, but with the cardinality of which is not known in advance, which is a frequent case in the field of Knowledge Representation. Bacchus [5] gave the similar argument in the context of probability logics, arguing that, while a domain may be finite, it is questionable if there is a fixed upper bound on its size, and he also pointed out that there are many domains, interesting for AI applications, that are not finite.

Second, we consider *infinite number of agents*. While this assumption is not of interest in probability logic, it was studied in epistemic logic. Halpern and Shore [26] pointed out that economies, when regarded as teams in a game, are often modeled as having infinitely many agents and that such modeling in epistemic logic is also convenient in the situations where the group of agents and its upper limit are not known *a priori*.

The semantics for our logic consists of Kripke models enriched with probability spaces. Each possible world contains a first-order structure, and each agent in each world is equipped with a set of accessible worlds and a finitely additive probability on measurable sets of worlds. In this article, we consider the most general semantics, with independent modalities for knowledge and probability. Nevertheless, in Section 5.2, we show how to modify the definitions and results of our logic to capture some interesting relationships between the modalities for knowledge and probability (previously considered in Reference [17]), especially the semantics in which agents assign probabilities only to the sets of worlds they consider possible.

The main result of this article is a sound and *strongly complete* ("every consistent set of sentences is satisfiable") axiomatization. The negative result of Wolter [49] shows that there is no finite way to axiomatize first-order common knowledge logics, and that infinitary axiomatizations are the best we can do (see Section 2.3). The same type of negative result for first-order probability logics is obtained by Abadi and Halpern [1]. We obtain completeness using infinitary rules of inference. Thus, formulas are finite, while only proofs are allowed to be (countably) infinite. We use a Henkin-style construction of saturated extensions of consistent theories. From the technical point of view, we modify some of our earlier developed methods presented in References [12, 13, 15, 16, 34, 39, 40, 43].[2] Although we use an alternative axiomatization for the epistemic part of logic (i.e., different from original axiomatization given in References [17, 24]), we prove that standard axioms are derivable in our system.

There are several papers on completeness of epistemic logics with common knowledge.

[2] For a detailed overview of the approach, we refer the reader to Reference [41].

In *propositional case*, a finitary axiomatization, which is *weakly complete* ("every consistent formula is satisfiable"), is obtained by Halpern and Moses [24] using a fixed-point axiom for common knowledge. However, strong completeness for any finitary axiomatization is impossible, due to lack of compactness (see Section 2.3). Strongly complete axiomatic systems are proposed in References [11, 45]. They contain an infinitary inference rule, similar to one of our rules, for capturing semantic relationship between the operators of group knowledge and common knowledge.

In *first-order case*, the set of valid formulas is not recursively enumerable [49] and, consequently, there is no complete finitary axiomatization. One way to overcome this problem is by including infinite formulas in the language as in Reference [46]. A logic with finite formulas, but an infinitary inference rule, is proposed in Reference [31], while a Genzen-style axiomatization with an inifinitary rule is presented in Reference [45]. However, a finitary axiomatization of monadic fragments of the logic, without function symbols and equality, is proposed in Reference [44].

Fagin and Halpern [17] proposed a joint frame for reasoning about knowledge and probability. Following the approach from Reference [18], they extended the propositional epistemic language with formulas that express linear combinations of probabilities, i.e., the formulas of the form $a_1 p(\varphi_1) + \cdots + a_k p(\varphi_k) \geq b$, where $a_1, .., a_k, b \in \mathbb{Q}$, $k \geq 1$. They proposed a finitary axiomatization and proved weak completeness using the small-model theorem. Our axiomatization technique is different. Since in the first-order case a complete finitary axiomatization is not possible, we use infinitary rules and we prove strong completeness using the Henkin-style method. We use unary probability operators and we axiomatize the probabilistic part of our logic following the techniques from Reference [41]. In particular, our logic incorporates the single-agent probability logic $LFOP_1$ from Reference [40]. However, our approach can be easily extended to include linear combinations of probabilities, similarly as it was done in References [14, 38].

Recently, some first-order logical systems have emerged dealing with common knowledge [7] and probabilistic beliefs [8, 9], but no proof system is presented for those logics. References [7, 8] consider a so-called "only knowing" notion with the basic idea that the beliefs of an agent are precisely those inferred from its knowledge base, providing semantics without axiomatization. Reference [7] introduces a first-order multiagent logic with "only knowing" and common knowledge. Its propositional part without "only knowing" operator is proven to be related to the $KC45_n$ logic ($K45$ modal logic with n agents and common knowledge [20, 30]). The authors study the interactions between the notions of common knowledge and only knowing, providing an account of the muddy children puzzle as logical implications of what is only known initially. References [8, 9] consider single agent only with probabilistic beliefs (degrees of belief) in the presence of noisy sensing and acting using situation calculus. Roughly speaking, the degree of belief in formula ϕ is a normalized sum of the weights of the worlds where ϕ holds and can be thought of as a subjective probability (initially introduced in Reference [6]). This notion can be expressed in our semantics, but we also allow different probability spaces across the worlds. In Reference [8], a first-order logic with "only knowing" and degrees of belief that capture the beliefs of a fully introspective knowledge base via "only knowing" operator is proposed. Reference [9] takes into consideration that many robotic applications require continuous domains and overcomes the limitation of the degrees of belief being restricted to discrete probability distributions in Reference [6] by incorporating continuous densities in its approach.

We point out that all the above mentioned logics do not support an infinite group of agents, so the group knowledge operator is defined as the conjunction of knowledge of individual agents. A weakly complete axiomatization for common knowledge with infinite number of agents (in non-probabilistic setting) is presented in Reference [26]. In our approach, the knowledge operators of groups and individual agents are related via an infinitary rule (RE from Section 3). It is used to overcome the non-compactness issue that appears in this extended framework with infinite number of agents (see Section 2.3 and Example 3.4).

To the best of our knowledge, this work provides the first logical framework with a proof system that combines reasoning about probability and epistemic reasoning in a first-order setting. As we already mentioned, the negative results from References [1, 49] show that it is not possible to obtain (even weakly) complete finite (recursive) axiomatization for this first-order framework. To achieve completeness, we propose an axiom system with infinitary rules of inference, and we combine and extend several existing approaches. We extend both the Henkin's method of constants for first-order logic [28] and our previously developed infinitary approaches for logics with possible world semantics [13, 14, 16, 39, 40] for establishing the strong completeness. We adopt the infinitary rule for connecting the epistemic operators of group knowledge and common knowledge from Reference [11] (the rule RC from Section 3) and generalize the standard Archimedean rule whose role is to controll the range of probability measures [41] (the rule RA from Section 3). In addition, we propose three novel infinitary rules: the rule RE, which relates group knowledge with knowledge of the members of the group, its probabilistic variant RPE, and RPC, which captures the semantic property of probabilistic common knowledge. Those three rules crucially contributed to the two novel aspects of this work:

- We propose the first axiomatization in the literature for a logic that includes the operator of probabilistic common knowledge.
- We propose the first strongly complete axiomatization of epistemic logic with infinite groups of agents.

We adapted all the infinitary rules of inference by considering premises and conclusions in the form of k-nested implications (Definition 2.3) to prove strong necessitation of knowledge operators (Theorem 4.2), which is crucial for the proof of Truth lemma (Lemma 5.4). k-nested implications are already used in probabilistic, epistemic, and temporal logics for obtaining various strong necessitation results [11, 32, 34].

The rest of the article is organized as follows: In Section 2, we introduce Syntax and Semantics. Section 3 provides the axiomatization of our logic system, followed by the proofs of its soundness. In Section 4, we prove several theorems, including Deduction theorem and Strong necessitation. The completeness result is proven in Section 5. In Section 6, we consider an extension of our logic by incorporating the *consistency condition* [17]. The concluding remarks are given in Section 7.

2 SYNTAX AND SEMATICS

In this section, we present the syntax and semantics of our logic, which we call $\mathcal{PCK}^{fo}$.[3] Since the main goal of this article is to combine the epistemic first-order logic with reasoning about probability, our language extends a first-order language with both epistemic operators, and the operators for reasoning about probability and probabilistic knowledge. We introduce the set of formulas based on this language and the corresponding possible world semantics, and we define the satisfiability relation.

2.1 Syntax

Let $[0,1]_{\mathbb{Q}}$ be the set of rational numbers from the real interval $[0,1]$, $\mathbb{N}$ the set of non-negative integers, $\mathcal{A}$ an at-most countable set of agents, and $\mathcal{G}$ a countable set of nonempty subsets of $\mathcal{A}$.

[3] $\mathcal{PCK}$ stands for "probabilistic common knowledge," while fo indicates that our logic is a first-order logic.

The language $\mathcal{L}_{PCK^{fo}}$ of the logic $\mathcal{P}CK^{fo}$ contains:

- a countable set of variables $Var = \{x_1, x_2, \ldots\}$,
- m-ary relation symbols $R_0^m, R_1^m, \ldots$ and function symbols $f_0^m, f_1^m, \ldots$ for every integer $m \geq 0$,
- Boolean connectives $\wedge$ and $\neg$, and the first-order quantifier $\forall$,
- unary modal knowledge operators K_i, E_G, C_G, for every $i \in \mathcal{A}$ and $G \in \mathcal{G}$,
- unary probability operator $P_{i,\geq r}$ and the operators for probabilistic knowledge E_G^r and C_G^r, where $i \in \mathcal{A}$, $G \in \mathcal{G}$, $r \in [0,1]_{\mathbb{Q}}$.

By the standard convention, constants are 0−ary function symbols. Terms and atomic formulas are defined in the same way as in the classical first-order logic.

Definition 2.1 (Formula). The set of formulas $For_{PCK^{fo}}$ is the least set containing all atomic formulas such that: if $\varphi, \psi \in \mathcal{F}or_{PCK^{fo}}$, then $\neg\varphi, \varphi \wedge \psi, K_i\varphi, E_G\varphi, C_G\varphi, E_G^r\varphi, C_G^r\varphi, P_{i,\geq r}\varphi \in \mathcal{F}or_{PCK^{fo}}$, for every $i \in \mathcal{A}$, $G \in \mathcal{G}$ and $r \in [0,1]_{\mathbb{Q}}$.

We use the standard abbreviations to introduce other Boolean connectives $\rightarrow$, $\vee$, and $\leftrightarrow$, the quantifier $\exists$, and the symbols $\bot$, $\top$. We also introduce the operator K_i^r (for $i \in \mathcal{A}$ and $r \in [0,1]_{\mathbb{Q}}$) in the following way: the formula $K_i^r\varphi$ abbreviates $K_i(P_{i,\geq r}\varphi)$.

The meanings of the operators of our logic are as follows:

- $K_i\varphi$ is read as *"agent i knows φ"* and $E_G\varphi$ as *"everyone in the group G knows φ."* The formula $C_G\varphi$ is read *"φ is common knowledge among the agents in G,"* which means that everyone (from G) knows φ, everyone knows that everyone knows φ, and so on.

  Example. The sentence *"everyone in the group G knows that if agent i doesn't know φ, then ψ is common knowledge in G,"* is written as
  $$E_G(\neg K_i\varphi \rightarrow C_G\psi).$$
- The probabilistic formula $P_{i,\geq r}\varphi$ says that *the probability that formula φ holds is at least r according to the agent i*.
- $K_i^r\varphi$ abbreviates the formula $K_i(P_{i,\geq r}\varphi)$. It means that *agent i knows that the probability of φ is at least r*.

  Example. Suppose that agent i considers two only possible scenarios for an event φ, and that each of these scenarios puts a different probability space on events. In the first scenario, the probability of φ is 1/2, and in the second one it is 1/4. Therefore, the agent knows that probability of φ is at least 1/4, i.e., $K_i(P_{i,\geq 1/4}\varphi)$.
- $E_G^r\varphi$ denotes that *everyone in the group G knows that the probability of φ is at least r*. Once $K_i^r\varphi$ is introduced, E_G^r is defined as a straightforward probabilistic generalization of the operator E_G.
- $C_G^r\varphi$ denotes that it is a *common knowledge in the group G that the probability of φ is at least r*. For a given threshold $r \in [0,1]_{\mathbb{Q}}$, C_G^r represents a generalization of non-probabilistic operator C_G.

  Example. The formula
  $$E_G^s(K_i(\exists x)\varphi(x) \wedge \neg C_G^r\psi)$$
  says that *everyone in the group G knows that the probability that both agent i knows that $\varphi(x)$ holds for some x, and that ψ is not common knowledge among the agents in G with probability at least r, is at least s.*

Note that the other types of probabilistic operators can also be introduced as abbreviations: $P_{i,<r}\varphi$ is $\neg P_{i,\geq r}\varphi$, $P_{i,\leq r}\varphi$ is $P_{i,\geq 1-r}\neg\varphi$, $P_{i,>r}\varphi$ is $\neg P_{i,\leq r}\varphi$, and $P_{i,=r}\varphi$ is $P_{i,\leq r}\varphi \wedge P_{i,\geq r}\varphi$.

Now, we define what we mean by a *sentence* and a *theory*. The following definition uses the notion *free variable*, which is defined in the same way as in the classical first-order logic.

Definition 2.2 (Sentence). A formula with no free variables is called a sentence. The set of all sentences is denoted by $Sent_{PCK^{fo}}$. A set of sentences is called *theory*.

Next, we introduce a special kind of formula in the implicative form, called *k-nested implications*, which will have an important role in our axiomatization.

Definition 2.3 (k-nested Implication). Let $\tau \in For_{PCK^{fo}}$ be a formula and let $k \in \mathbb{N}$. Let $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_k)$ be a sequence of k formulas, and $\mathbf{X} = (X_1, \ldots, X_k)$ a sequence of knowledge operators from $\{\mathcal{K}_i \mid i \in \mathcal{A}\}$. The *k-nested implication* formula $\Phi_{k,\boldsymbol{\theta},\mathbf{X}}(\tau)$ is defined inductively, as follows:

$$\Phi_{k,\boldsymbol{\theta},\mathbf{X}}(\tau) = \begin{cases} \theta_0 \to \tau, \ \ k = 0 \\ \theta_k \to X_k \Phi_{k-1,\boldsymbol{\theta}_{j=0}^{k-1},\mathbf{X}_{j=0}^{k-1}}(\tau), \ \ k \geq 1. \end{cases}$$

For example, if $\mathbf{X} = (K_a, K_b, K_c)$, $a, b, c \in \mathcal{A}$, then

$$\Phi_{3,\boldsymbol{\theta},\mathbf{X}}(\tau) = \theta_3 \to K_c(\theta_2 \to K_b(\theta_1 \to K_a(\theta_0 \to \tau))).$$

Formulas of this form are used to formulate infinitary inference rules that are essential in the axiomatization presented in Section 3. The need of k-nested implications in those rules comes directly from our completeness proof where they give a form of deep inference that is very similar to nested sequents [10]. Deep inference refers to deductive systems in which rules can not only be applied to outermost connectives but also be deep inside formulas.

The structure of these k-nested implications is shown to be especially convenient for the proof of the Deduction theorem (Theorem 4.1) and the Strong necessitation theorem (Theorem 4.2), as we discuss in Remark 2 and Remark 3.

2.2 Semantics

The semantic approach for PCK^{fo} extends the classical possible-worlds model for epistemic logics, with probabilistic spaces.

Definition 2.4 (PCK^{fo} Model). A PCK^{fo} model is a *Kripke structure for knowledge and probability* that is represented by a tuple

$$M = (S, D, I, \mathcal{K}, \mathcal{P}),$$

where:

- S is a nonempty set of *states* (or *possible worlds*);
- D is a nonempty domain;
- I associates an interpretation $I(s)$ with each state s in S such that for all $i \in \mathcal{A}$ and all $k, m \in \mathbb{N}$:
  - $-I(s)(f_k^m)$ is a function from D^m to D,
  - $-$for each $s' \in S$, $I(s')(f_k^m) = I(s)(f_k^m)$,
  - $-I(s)(R_k^m)$ is a subset of D^m;
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathcal{A}\}$ is a set of binary relations on S. We denote $\mathcal{K}_i(s) \stackrel{def}{=} \{t \in s \mid (s,t) \in \mathcal{K}_i\}$, and write $s\mathcal{K}_i t$ if $t \in \mathcal{K}_i(s)$;
- $\mathcal{P}$ associates to every agent $i \in \mathcal{A}$ and every state $s \in S$ a probability space $\mathcal{P}(i,s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$, such that
  - $-S_{i,s}$ is a non-empty subset of S,
  - $-\chi_{i,s}$ is an algebra of subsets of $S_{i,s}$, whose elements are called *measurable sets*; and

— $\mu_{i,s} : \chi_{i,s} \rightarrow [0,1]$ is a finitely additive probability measure, i.e.,
  * $\mu_{i,s}(S_{i,s}) = 1$, and
  * $\mu_{i,s}(A \cup B) = \mu_{i,s}(A) + \mu_{i,s}(B)$ if $A \cap B = \emptyset, A, B \in \chi_{i,s}$.

In the previous definition, we assume that the domain is fixed (i.e., the domain is same in all the worlds) and that the terms are rigid, i.e., for every model their meanings are the same in all worlds. Intuitively, the first assumption means that it is common knowledge that objects exist. Note that the second assumption implies that it is common knowledge which object a constant designates. As it is pointed out in Reference [44], the first assumption is natural for all those application domains that deal not with knowledge about the existence of certain objects, but rather with knowledge about facts. Also, the two assumptions allow us to give semantics of probabilistic formulas, which is similar to the objectual interpretation for first-order modal logics [21].

Note that those standard assumptions for modal logics are essential to ensure validity of all first-order axioms. For example, if the terms are not rigid, the classical first-order axiom

$$\forall \varphi(x) \rightarrow \varphi(t),$$

where the term t is free for x in φ, would not be valid (an example is given in Reference [22]). Similarly, Barcan formula (axiom FO3 in Section 3) holds only for fixed domain models.

For a model $M = (S, D, I, \mathcal{K}, \mathcal{P})$ to be a PCK^{fo}, the notion of *variable valuation* is defined in the usual way: a variable valuation v is a function that assigns the elements of the domain to the variables, i.e., $v : Var \rightarrow D$. If v is a valuation, then $v[d/x]$ is a valuation identical to v, with the exception that $v[d/x](x) = d$.

Definition 2.5 (Value of a Term). The value of a term t in a state s with respect to v, denoted by $I(s)(t)_v$, is defined in the following way:

- if $t \in Var$, then $I(s)(t)_v = v(t)$,
- if $t = F_j^k(t_1, \ldots, t_k)$, then $I(s)(t)_v = I(s)(F_j^k)(I(s)(t_1)_v, \ldots, I(s)(t_k)_v)$.

The next definition will use the following knowledge operators, which we introduce in the inductive way:

- $(E_G)^1\varphi = E_G\varphi$;
- $(E_G)^{m+1}\varphi = E_G((E_G)^k\varphi)$, $m \in \mathbb{N}$;
- $(F_G^r)^0\varphi = \top$;
- $(F_G^r)^{m+1}\varphi = E_G^r(\varphi \wedge (F_G^r)^m\varphi)$, $m \in \mathbb{N}$.

Now, we define *satisfiability* of formulas from in the states of introduced models.

Definition 2.6 (Satisfiability Relation). Satisfiability of formula φ in a state $s \in S$ of a model M, under a valuation v, denoted by

$$(M, s, v) \models \varphi,$$

is defined in the following way:

(1) $(M, s, v) \models R_j^k(t_1, \ldots, t_k)$ iff $(I(s)(t_1)_v, \ldots, I(s)(t_k)_v) \in I(s)(R_j^k)$
(2) $(M, s, v) \models \neg\varphi$ iff $(M, s, v) \not\models \varphi$
(3) $(M, s, v) \models \varphi \wedge \psi$ iff $(M, s, v) \models \varphi$ and $(M, s, v) \models \psi$
(4) $(M, s, v) \models (\forall x)\varphi$ iff for every $d \in D$, $(M, s, v[d/x]) \models \varphi$
(5) $(M, s, v) \models K_i\varphi$ iff $(M, t, v) \models \varphi$ for all $t \in \mathcal{K}_i(s)$
(6) $(M, s, v) \models E_G\varphi$ iff $(M, s, v) \models K_i\varphi$ for all $i \in G$
(7) $(M, s, v) \models C_G\varphi$ iff $(M, s, v) \models (E_G)^m\varphi$ for every $m \in \mathbb{N}$
(8) $(M, s, v) \models P_{i,\geq r}\varphi$ iff $\mu_{i,s}(\{t \in S_{i,s} \mid (M, t, v) \models \varphi\}) \geq r$

(9) $(M, s, v) \models E^r_G\varphi$ iff $(M, s, v) \models K^r_i\varphi$ for all $i \in G$
(10) $(M, s, v) \models C^r_G\varphi$ iff $(M, s, v) \models (F^r_G)^m\varphi$ for every $m \in \mathbb{N}$

Note that the satisfiability relation $\models$ defined in Definition 2.6 is a *partial relation*, i.e., it is not in general defined for all formulas. Indeed, a formula of the form $P_{i,\geq r}\varphi$ can be evaluated only under assumption that the corresponding set of states

$$[\varphi]^v_{i,s} = \{s \in S_{i,s} \mid (M, s, v) \models \varphi\}$$

is a measurable set. To keep the satisfiability relation *total*, i.e., well-defined for all the formulas, in this article, we consider only the models in which all those sets are measurable.

Definition 2.7 (Measurable Model). A model $M = (S, D, I, \mathcal{K}, \mathcal{P})$ is a *measurable model* if

$$[\varphi]^v_{i,s} \in \chi_{i,s},$$

for every formula φ, valuation v, state s, and agent i. We denote the class of all these models as $\mathcal{M}^{MEAS}_{\mathcal{A}}$.

Remark 1. The semantic definition of the probabilistic common knowledge operator C^r_G from the last item of Definition 2.6 is first proposed by Fagin and Halpern in Reference [17], as a generalization of the operator C_G regarded as the infinite conjunction of all degrees of group knowledge. It is important to mention that this is not the only proposal for generalizing the non-probabilistic case. Monderer and Samet [35] proposed a more intuitive definition, where probabilistic common knowledge is semantically equivalent to the infinite conjunction of the formulas $E^r_G\varphi, (E^r_G)^2\varphi, (E^r_G)^3\varphi \ldots$ Although both are legitimate probabilistic generalizations, in this article, we accept the definition of Fagin and Halpern [17], who argued that their proposal seems more adequate for the analysis of problems such as probabilistic coordinated attack and Byzantine agreement protocols [27]. As we point out in the Conclusion, our axiomatization approach can be easily modified to capture the definition of Monderer and Samet.

If $(M, s, v) \models \varphi$ holds for every valuation v, then we write $(M, s) \models \varphi$. If $(M, s) \models \varphi$ for all $s \in S$, then we write $M \models \varphi$.

Definition 2.8 (Satisfiability of Sentences). A sentence φ is *satisfiable* if there is a state s in some model M such that $(M, s) \models \varphi$. A set of sentences T is satisfiable if there exists a state s in a model M such that $(M, s) \models \varphi$ for each $\varphi \in T$. A sentence φ is *valid* if $\neg\varphi$ is not satisfiable.

Note that in the previous definition the satisfiability of sentences does not depend on a valuation, since they do not contain any free variable.

Observe that if φ is a sentence, then the set $[\varphi]^v_{i,s}$ does not depend on v; thus, we relax the notation by denoting it by $[\varphi]_{i,s}$. Also, we write $\mu_{i,s}([\varphi])$ instead of $\mu_{i,s}([\varphi]_{i,s})$.

2.3 Axiomatization Issues

At the end of this section, we analyze two common characteristics of epistemic logics and probability logics, which have impacts on their axiomatizations.

The first one is the *non-compactness* phenomenon—there are unsatisfiable sets of formulas such that all their finite subsets are satisfiable. The existence of such sets in epistemic logic is a consequence of the fact that the common knowledge operator C_G can be semantically seen as an infinite conjunction of all the degrees of the group knowledge operator E_G, which leads to the example

$$\{(E_G)^m\varphi \mid m \in \mathbb{N}\} \cup \{\neg C_G\varphi\}.$$

In real-valued probability logics, a standard example of unsatisfiable set whose finite subsets are all satisfiable is

$$\{P_{i,\geq 1-\frac{1}{n}}\varphi \mid m \in \mathbb{N}\} \cup \{\neg P_{i,\neq 1}\varphi\},$$

where φ is a satisfiable sentence that is not valid.

Apart from those two classical examples of non-compactness of epistemic and probability logics, this specific work faces three additional sources of non-compactness. The first two are due to the fact that there might exist an infinite group of agents $G \in \mathcal{G}$:

$$\{K_i\varphi \mid i \in \mathbb{G}\} \cup \{\neg E_G\varphi\}, \{K_i^r\varphi \mid i \in \mathbb{G}\} \cup \{\neg E_G^r\varphi\},$$

while the third is due to the presence of operators of probabilistic common knowledge:

$$\{(F_G^r)^m\varphi \mid m \in \mathbb{N}\} \cup \{\neg C_G^r\varphi\}.$$

One significant consequence of non-compactness is that there is no finitary axiomatization that is strongly complete [48], i.e., simple completeness is the most one can achieve.

In the first-order case, the situation is even worse. Namely, the set of valid formulas is not *recursively enumerable*, neither for first-order logic with common knowledge [49] nor for first-order probability logics [1] (moreover, even their monadic fragments suffer from the same drawback [41, 49]). This means that there is no finitary axiomatization that could be (even simply) complete. An approach for overcoming this issue, proposed by Wolter [49] and Ognjanovic and Raskovic [39], is to consider infinitary logics as the only interesting alternative.

In this article, we introduce the axiomatization with *ω-rules* (inference rules with countably many premises) [11, 40]. This allows us to keep the object language countable and to move infinity to meta language only: The formulas are finite, while only proofs are allowed to be infinite.

3 THE AXIOMATIZATION $Ax_{PCK^{fo}}$

In this section, we introduce the axiomatic system for the logic PCK^{fo}, denoted by $Ax_{PCK^{fo}}$. It consists of the following axiom schemata and rules of inference:

I First-order axioms and rules

Prop. All instances of tautologies of the propositional calculus

MP. $\dfrac{\varphi, \varphi \to \psi}{\psi}$ (Modus Ponens)

FO1. $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is not a free variable un φ

FO2. $\forall\varphi(x) \to \varphi(t)$, where $\varphi(t)$ is the result of substitution of all free occurrences of x in $\varphi(x)$ by a term t that is free for x in $\varphi(x)$

FO3. $\forall x K_i\varphi(x) \to K_i\forall x\varphi(x)$ (Barcan formula)

FOR. $\dfrac{\varphi}{\forall x\varphi}$

II Axioms and rules for reasoning about knowledge

AK. $(K_i\varphi \wedge K_i(\varphi \to \psi)) \to K_i\psi$, $i \in G$ (Distribution Axiom)

RK. $\dfrac{\varphi}{K_i\varphi}$ (Knowledge Necessitation)

AE. $E_G\varphi \to K_i\varphi$, $i \in G$

RE. $\dfrac{\{\Phi_{k,\theta,\mathrm{X}}(K_i\varphi) \mid i \in G\}}{\Phi_{k,\theta,\mathrm{X}}(E_G\varphi)}$

AC. $C_G\varphi \to (E_G)^m\varphi, m \in \mathbb{N}$

RC. $\dfrac{\{\Phi_{k,\theta,\mathrm{X}}((E_G)^m\varphi) \mid m \in \mathbb{N}\}}{\Phi_{k,\theta,\mathrm{X}}(C_G\varphi)}$

III Axioms and rule for reasoning about probabilities

P1. $P_{i,\geq 0}\varphi$
P2. $P_{i,\leq r}\varphi \rightarrow P_{i,<t}\varphi,\ t > r$
P3. $P_{i,<t}\varphi \rightarrow P_{i,\leq t}\varphi$
P4. $(P_{i,\geq r}\varphi \wedge P_{i,\geq t}\psi \wedge P_{i,\geq 1}\neg(\varphi \wedge \psi)) \rightarrow P_{i,\geq min(1,r+t)}(\varphi \vee \psi)$
P5. $(P_{i,\leq r}\varphi \wedge P_{i,<t}\varphi) \rightarrow P_{i,<r+t}(\varphi \vee \psi),\ r + t \leq 1$
RP. $\dfrac{\varphi}{P_{i,\geq 1}\varphi}$ (Probabilistic Necessitation)
RA. $\dfrac{\{\Phi_{k,\theta,\mathrm{X}}(P_{i,\geq r-\frac{1}{m}}\varphi) \mid m \geq \frac{1}{r},\ m \in \mathbb{N}\ \}}{\Phi_{k,\theta,\mathrm{X}}(P_{i,\geq r}\varphi)}$, $r \in (0,1]_{\mathbb{Q}}$ (Archimedean rule)

IV Axioms and rules for reasoning about probabilistic knowledge

APE. $E^r_G\varphi \rightarrow K^r_i\varphi,\ i \in G$
RPE. $\dfrac{\{\Phi_{k,\theta,\mathrm{X}}(K^r_i\varphi) \mid i \in G\}}{\Phi_{k,\theta,\mathrm{X}}(E^r_G\varphi)}$
APC. $C^r_G\varphi \rightarrow (F^r_G)^m\varphi,\ m \in \mathbb{N}$
RPC. $\dfrac{\{\Phi_{k,\theta,\mathrm{X}}(F^r_G)^m\varphi) \mid m \in \mathbb{N}\}}{\Phi_{k,\theta,\mathrm{X}}(C^r_G\varphi)}$

The given axioms and rules are divided into four groups, according to the type of reasoning. The first group contains the standard axiomatization for first-order logic and, in addition, a variant of the well-known axiom for modal logics, called Barcan formula. It is proved that Barcan formula holds in the class of all first-order fixed domain modal models, and that it is independent from the other modal axioms [29, 30]. The second group contains axioms and rules for epistemic reasoning. AK and RK are classical Distribution axiom and Necessitation rule for the knowledge operator. The axiom AE and the rule RE are novel; they properly relate the knowledge operators and the operator of group knowledge E_G, regardless of the cardinality of the group G. Similarly, AC and RC properly relate the operators E_G and C_G. The infinitary rule RC is a generalization of the rule $InfC$ from Reference [11]. The third group contains a multi-agent variant of a standard axiomatization for reasoning about probability [41]. The infinitary rule RA is a variant of the so-called Archimedean rule, generalized by incorporating the k-nested implications in a similar way as it has been done in Reference [34] in purely probabilistic settings. This rule informally says that if probability of a formula is considered by an agent i to be arbitrarily close to some number r, then, according to the agent i, the probability of the fomula must be equal to r. The last group consists of novel axioms and rules that allow reasoning about probabilistic knowledge. They properly capture the semantic relationship between the operators K^r_i, E^r_G, F^r_G, and C^r_G, and they are in spirit similar to the last four axioms and rules from the second group.

Altogether, our axiomatization contains five infinitary rules of inference; they directly correspond to the five sources of non-compactness of our logic, identified in Section 2.3—each of them can be used to formally prove the inconsistency of the corresponding example of an unsatisfiable set (we illustrate that fact by Example 3.4, for the rule RE). Note that those sets would be consistent under any finitary axiomatization.

Note that we use the structure of these k-nested implications in all of our infinitary inference rules. As we have already mentioned, the reason is that this form allows us to prove the Deduction theorem and Strong necessitation theorem. Note that by choosing $k = 0$, $\theta_0 = \top$ in the inference rules RE, RC, RPE, RPC, we obtain the intuitive forms of the rules:

$$\frac{\{K_i\varphi \mid i \in G\}}{E_G\varphi}, \frac{\{(E_G)^m\varphi \mid m \geq 1\}}{C_G\varphi}, \frac{\{K_i^r\varphi \mid i \in G\}}{E_G^r\varphi}, \frac{\{(F_G^r)^m\varphi \mid m \geq 0\}}{C_G^r\varphi}.$$

Next, we define some basic notions of proof theory.

Definition 3.1. A formula φ is a *theorem*, denoted by $\vdash \varphi$, if there is an at-most countable sequence of formulas $\varphi_0, \varphi_1, \ldots, \varphi_{\lambda+1}$ (λ is a finite or countable ordinal[4]) of formulas from $For_{PCK^{fo}}$, such that $\varphi_{\lambda+1} = \varphi$, and every φ_i is an instance of some axiom schemata or is obtained from the preceding formulas by an inference rule.

A formula φ is *derivable from* a set T of formulas ($T \vdash \varphi$) if there is an at-most countable sequence of formulas $\varphi_0, \varphi_1, \ldots, \varphi_{\lambda+1}$ (λ is a finite or countable ordinal) such that $\varphi_{\lambda+1} = \varphi$, and each φ_i is an instance of some axiom schemata or a formula from the set T, or it is obtained from the previous formulas by an inference rule, with the exception that the premises of the inference rules RK and RP must be theorems. The corresponding sequence of formulas is a *proof* for φ from T.

A set of formulas T is *deductively closed* if it contains all the formulas derivable from T, i.e., $\varphi \in T$ whenever $T \vdash \varphi$.

Obviously, a formula is a theorem iff it is derivable from the empty set. Now, we introduce the notions of consistency and maximal consistency.

Definition 3.2. A set T of formulas is *inconsistent* if $T \vdash \varphi$ for every formula φ, otherwise it is *consistent.* A set T of formulas is *maximal consistent* if it is consistent and each proper superset of T is inconsistent.

It is easy to see that T is inconsistent iff $T \vdash \bot$.

In the proof of completeness theorem, we will use a special type of maximal consistent set, called saturated sets.

Definition 3.3. A set T of formulas is *saturated* iff it is maximal consistent and the following condition holds:

if $\neg(\forall x)\varphi(x) \in T$, then there is a term t such that $\neg\varphi(t) \in T$.

Note the notions of *deductive closeness, maximal consistency,* and *saturates sets* are defined for formulas, but they can be defined for *theories* (sets of sentences) in the same way. We omit the formal definitions here, since they would have the identical form as the ones above, but we will use the mentioned notions in the following sections.

Now, we show an example of how the infinitary rules are used in practice.

Example 3.4. Let us show that the set $T = \{K_i\varphi \mid i \in G\} \cup \{\neg E_G\varphi\}$ of formulas saying that each member of some (possibly infinite) group G knows φ and that the whole group G does not know φ is an inconsistent set with respect to our axiomatization:

(1) $T \vdash \neg E_G\varphi$, since $\neg E_G\varphi \in T$
(2) $T \vdash K_i\varphi$, since $K_i\varphi \in T$, for all $i \in G$
(3) $T \vdash E_G\varphi$, by RE for $k = 0$ and $\theta = \top$
(4) $T \vdash \bot$, from (1) and (3).

[4]I.e., the length of a proof is an at-most countable *successor* ordinal.

The following result shows that the proposed axioms from $Ax_{PCK^{fo}}$ are valid, and the inference rules preserve validity.

Theorem 3.5 (Soundness). *The axiomatic system $Ax_{PCK^{fo}}$ is sound with respect to the class of PCK^{fo} models.*

Proof. The soundness of the propositional part follows directly from the fact that interpretation of $\wedge$ and $\neg$ in the definition of $\models$ relation is the same as in the propositional calculus. The proofs for FO1. and FOR. are standard.

AE. AC., and APC. follow immediately from the semantics of operators E_G, C_G, and C_G^r.

FO2. Let $(M,s) \models (\forall x)\varphi(x)$. Then $(M,s,v) \models (\forall x)\varphi(x)$ for every valuation v. Note that for every v, among all valuations there must be a valuation v' such that $v'(s)(x) = d = I(s)(t)_v$ and $(M,s,v') \models \varphi(x)$. From the equivalence $(M,s,v') \models \varphi(x)$ iff $(M,s,v) \models \varphi(t)$, we obtain that $(M,s,v) \models \varphi(t)$ holds for every valuation. Thus, every instance of FO2 is valid.

FO3. (Barcan formula) Suppose that $(M,s) \models (\forall x)K_i\varphi(x)$, i.e., for each evaluation v, $(M,s,v) \models (\forall x)K_i\varphi(x)$. Then for each valuation v and every $d \in D$, $(M,s,v[d/x]) \models K_i\varphi(x)$. Therefore, for every v and d and every $t \in \mathcal{K}_i(s)$, we have $(M,t,v[d/x]) \models \varphi(x)$. Thus, for every $t \in K_i(s)$ and every valuation v, $(M,t,v) \models (\forall x)\varphi(x)$. Finally, since for every $t \in K_i(s)$, $(M,t) \models (\forall x)\varphi(x)$, we have $(M,s) \models K_i(\forall x)\varphi(x)$.

RC. We will prove by induction on k that if $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}((E_G)^m\varphi)$, for all $m \in \mathbb{N}$, then also $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}(C_G\varphi)$, for each state s and valuation v of any Kripke structure M:

Induction base $k = 0$. Let $(M,s,v) \models \theta_0 \to (E_G)^m\varphi$, for all $m \in \mathbb{N}$. Assume that it is not $(M,s,v) \models \theta_0 \to C_G\varphi$, i.e.,

$$(M,s,v) \models \theta_0 \wedge \neg C_G\varphi. \tag{3.1}$$

Then, $(M,s,v) \models (E_G)^m\varphi$, for all $m \in \mathbb{N}$, and therefore $(M,s,v) \models C_G\varphi$ (by the definition of the satisfiability relation), which contradicts Equation (3.1).

Inductive step. Let $(M,s,v) \models \Phi_{k+1,\boldsymbol{\theta},\mathbf{X}}((E_G)^m\varphi)$, for all $m \in \mathbb{N}$ where $\boldsymbol{\theta} = (\theta_0,\ldots,\theta_{k+1})$.

Suppose $X_{k+1} = K_i$ for some $i \in \mathcal{A}$, i.e., $(M,s,v) \models \theta_{k+1} \to K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}((E_G)^m\varphi)$, for all $m \in \mathbb{N}$. Assume the opposite, that $(M,s,v) \not\models \Phi_{k+1,\boldsymbol{\theta},\mathbf{X}}(C_G\varphi)$, i.e., $(M,s,v) \models \theta_{k+1} \wedge \neg K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(C_G\varphi)$. Then also $(M,s,v) \models K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}((E_G)^m\varphi)$, for all $m \in \mathbb{N}$, so for every state $t \in \mathcal{K}_i(s)$, we have that $(M,t,v) \models \Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}((E_G)^m\varphi)$, for all $m \in \mathbb{N}$, and by the induction hypothesis $(M,t,v) \models \Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(C_G\varphi)$. Therefore, $(M,s,v) \models K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(C_G\varphi)$, leading to a contradiction.

RA. We prove the soundness of this rule by induction on k, i.e., if $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}(P_{i,\geq r-\frac{1}{m}}\varphi)$ for every $m \in \mathbb{N}$, $m \geq \frac{1}{r}$ and $r > 0$, given some model M, state s and valuation v, then $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}(P_{i,\geq r}\varphi)$.

Induction base $k = 0$. This case follows by the properties of the real numbers.

Inductive step. Let $(M,s,v) \models \Phi_{k+1,\boldsymbol{\theta},\mathbf{X}}(P_{i,\geq r-\frac{1}{m}}\varphi)$ and $X_{k+1} = K_i$ for some $i \in \mathcal{A}$, i.e., $(M,s,v) \models \theta_{k+1} \to K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r-\frac{1}{m}}\varphi)$ for every $m \in \mathbb{N}$, $m \geq \frac{1}{r}$. Assume the opposite, that $(M,s,v) \not\models \Phi_{k+1,\boldsymbol{\theta},\mathbf{X}}(P_{i,\geq r}\varphi)$. Then, $(M,s,v) \models \theta_{k+1} \wedge \neg K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r}\varphi)$, so $(M,s,v) \models K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r-\frac{1}{m}}\varphi)$ for every $m \in \mathbb{N}$, $m \geq \frac{1}{r}$. Therefore, for every state $t \in \mathcal{K}_i(s)$: $(M,t,v) \models \Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r-\frac{1}{m}}\varphi)$ and $(M,t,v) \models \Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r\varphi})$ by the induction hypothesis, so $(M,s,v) \models K_i\Phi_{k,\boldsymbol{\theta}_{j=0}^k,\mathbf{X}_{j=0}^k}(P_{i,\geq r}\varphi)$, which is a contradiction.

RPC. Now, we show that rule RPC preserves validity by induction on k.

Let us prove the implication: if $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}((F_G^r)^m\varphi)$, for all $m \in \mathbb{N}$, and PCK_∞^{fo}-models M, then also $(M,s,v) \models \Phi_{k,\boldsymbol{\theta},\mathbf{X}}(C_G^r\varphi)$, for each state s in M:

Induction base $k = 0$. Suppose $(M, s, v) \models \theta_0 \to (F^r_G)^m\varphi$ for all $m \in \mathbb{N}$. If it is not $(M, s, v) \models \theta_0 \to C^r_G\varphi$, i.e., $(M, s, v) \models \theta_0 \wedge \neg C^r_G\varphi$, then $(M, s, v) \models (F^r_G)^m\varphi$, for all $m \in \mathbb{N}$. Therefore, $(M, s, v) \models C^r_G\varphi$, which is a contradiction.

Inductive step. Let $(M, s, v) \models \Phi_{k+1,\theta,\mathrm{X}}((F^r_G)^m\varphi)$, for all $m \in \mathbb{N}$.

Suppose $X_{k+1} = K_i, i \in \mathcal{A}$, i.e., $(M, s, v) \models \theta_{k+1} \to K_i\Phi_{k,\theta^k_{j=0},\mathrm{X}^k_{j=0}}((F^r_G)^m\varphi)$, for all $m \in \mathbb{N}$. If $s \not\models \Phi_{k+1,\theta,\mathrm{X}}(C^r_G\varphi)$, i.e., $(M, s, v) \models \theta_{k+1} \wedge \neg K_i\Phi_{k,\theta^k_{j=0},\mathrm{X}^k_{j=0}}(C^r_G\varphi)$ (*), then $(M, s, v) \models K_i\Phi_{k,\theta^k_{j=0},\mathrm{X}^k_{j=0}}((F^r_G)^m\varphi)$, for all $m \in \mathbb{N}$. So for each $t \in \mathcal{K}_i(s)$, we have $(M, t, v) \models \Phi_{k,\theta,\mathrm{X}}((F^r_G)^m\varphi)$. By the induction hypothesis on k it follows that $(M, t, v) \models \Phi_{k,\theta^k_{j=0},\mathrm{X}^k_{j=0}}(C^r_G\varphi)$. But then $(M, s, v) \models K_i\Phi_{k,\theta^k_{j=0},\mathrm{X}^k_{j=0}}(C^r_G\varphi)$, which contradicts (*). □

4 SOME THEOREMS OF PCK^{fo}

In this section, we prove several theorems. Some of them will be useful in proving the completeness of the axiomatization $Ax_{PCK^{fo}}$. We start with the deduction theorem. Since we will frequently use this theorem, we will not always explicitly mention it in the proofs.

Theorem 4.1 (Deduction Theorem). *If T is a theory and φ, ψ are sentences, then $T \cup \{\varphi\} \vdash \psi$ implies $T \vdash \varphi \to \psi$.*

Proof. We use the transfinite induction on the length of the proof of ψ from $T \cup \{\varphi\}$. The case $\psi = \varphi$ is obvious; if ψ is an axiom, then $\vdash \psi$, so $T \vdash \psi$, and therefore $T \vdash \varphi \to \psi$. If ψ was obtained by rule RK, i.e., $\psi = K_i\varphi$ where φ is a theorem, then $\vdash K_i\varphi$ (by R2); that is, $\vdash \psi$, so $T \vdash \varphi \to \psi$. The reasoning is analogous for cases of other inference rules that require a theorem as a premise. Now, we consider the case where ψ was obtained by rule RPC. The proof for the other infinitary rules is similar.

Let $T, \varphi \vdash \{\Phi_{k,\theta,\mathrm{X}}((F^r_G)^m\eta) \mid m \in \mathbb{N}\} \vdash \psi$ where $\psi = \Phi_{k,\theta,\mathrm{X}}(C^r_G\eta), \theta = (\theta_0, \theta_1 \dots, \theta_k), k \geq 1$. Then,

$T \vdash \varphi \to \Phi_{k,\theta,\mathrm{X}}((F^r_G)^m\eta)$, for all $m \in \mathbb{N}$, by the induction hypothesis.

Suppose $X_k = K_i$, for some $i \in \mathcal{A}$.

$T \vdash \varphi \to (\theta_k \to K_i\Phi_{k-1,\theta^{k-1}_{j=0},\mathrm{X}^{k-1}_{j=0}}((F^r_G)^m\eta))$, by the definition of Φ_k

$T \vdash (\varphi \wedge \theta_k) \to K_i\Phi_{k-1,\theta^{k-1}_{j=0},\mathrm{X}^{k-1}_{j=0}}((F^r_G)^m\eta)$, by the propositional tautology

$$(p \to (q \to r)) \longleftrightarrow ((p \wedge q) \to r). \tag{4.1}$$

Let $\overline{\theta} = (\theta_0, \dots, \theta_{k-1}, \varphi \wedge \theta_k)$. Then, we have:

$T \vdash \overline{\theta_k} \to K_i\Phi_{k-1,\overline{\theta}^{k-1}_{j=0},\mathrm{X}^{k-1}_{j=0}}((F^r_G)^m\eta)$, for all $m \in \mathbb{N}$

$T \vdash \Phi_{k,\overline{\theta},\mathrm{X}}((F^r_G)^m\eta)$, for all $m \in \mathbb{N}$

$T \vdash \Phi_{k,\overline{\theta},\mathrm{X}}(C^r_G\eta)$ by RPC

$T \vdash (\varphi \wedge \theta_k) \to K_i\Phi_{k-1,\theta^{k-1}_{j=0},\mathrm{X}^{k-1}_{j=0}}(C^r_G\eta)$

$T \vdash \varphi \to (\theta_k \to K_i\Phi_{k-1,\theta^{k-1}_{j=0},\mathrm{X}^{k-1}_{j=0}}(C^r_G\eta)$, by the tautology (4.1)

$T \vdash \varphi \to \Phi_{k,\theta,\mathrm{X}}(C^r_G\eta)$

$T \vdash \varphi \to \psi$.

The case $k = 0$ follows in a similar way. □

Remark 2 (Implicative form of the Infinitary Rules). Note that the proof of the Deduction theorem relies on the fact that the infinitary rules of inference are given in the implicative form, i.e., all the formulas in the premises and the conclusion of (every instance of) a rule are implications with the same antecedent. This is a standard technical solution that ensures that the Deduction theorem can be proved using the transfinite induction on the length of the inference. They are used, among others, in probabilistic, temporal, and epistemic logic, in the approaches when strong completeness theorems are obtained using infinitary rules to overcome non-compactness issues of the logics [2, 11, 13, 33, 38].

Let us explain the intuition behind the use of implicative forms. Suppose that the rule

$$\frac{\{\psi_m \mid m \geq 0\}}{\psi} \tag{4.2}$$

intuitively captures some specific non-compactness issue (e.g., in this article, we consider $\frac{\{(F_G^r)^m\varphi \ \mid\ m\geq 0\}}{C_G^r\varphi}$ to describe probabilistic common knowledge operator).

In the inductive proof of the Deduction theorem, in the case when $T, \theta \vdash \psi$ is obtained by an application of the rule (4.2), we need to use the induction hypothesis $T, \theta \vdash \psi_m \ \Rightarrow\ T \vdash \theta \to \psi_m$, for every $m \geq 0$. Then the implicative form (with antecedent θ) is an obstacle to directly apply the rule (4.2) on the set $\{\theta \to \psi_m \mid m \geq 0\}$ to infer $T \vdash \theta \to \psi$. For that reason, the rule (4.2) is relaxed to include all inferences of the form

$$\frac{\{\varphi \to \psi_m \mid m \geq 0\}}{\varphi \to \psi} \tag{4.3}$$

for every formula φ. This modification allows us to conclude $T \vdash (\theta \wedge \varphi) \to \psi_m$ from $T, \theta \vdash \varphi \to \psi_m$ in the induction step, using the tautology (4.1) for every $m \geq 0$. Then, we can apply the instance $\frac{\{(\theta\wedge\varphi)\to\psi_m \ \mid\ m\geq 0\}}{(\theta\wedge\varphi)\to\psi}$ of (4.3) and finally conclude $\theta \to (\varphi \to \psi)$ using (4.1).

Note that this proof strategy requires only the implicative form of the rule, while nesting of knowledge operators is not used in the proof of the Deduction theorem. However, the nesting of those operators is necessary for the proof of the Strong necessitation theorem (Theorem 4.2), and we discuss its use in Remark 3.

Next, we prove several results about the purely epistemic part of our logic. First, we show that the strong variant of necessitation for knowledge operator is a consequence of the axiomatization $Ax_{PCK^{fo}}$. This theorem will have an important role in the proof of completeness theorem, in the construction of the canonical model.

First, we need to introduce some notation. For a given set of formulas T and $i \in \mathcal{A}$, we define the set K_iT as the set of all formulas $K_i\varphi$, where φ belongs to T, i.e.,

$$K_iT = \{K_i\varphi \mid \varphi \in T\}.$$

THEOREM 4.2 (STRONG NECESSITATION). *If T is a theory and $T \vdash \varphi$, then $K_iT \vdash K_i\varphi$, for all $i \in \mathcal{A}$.*

PROOF. Let $T \vdash \varphi$. We will prove $K_iT \vdash K_i\varphi$ using the transfinite induction on the length of proof of $T \vdash \varphi$. Here, we will only consider the application of rules FOR and RPC, while the cases when we apply the other infinitary rules are similar to the proof for RCP.

(1) Suppose that $T \vdash \varphi$, where $\varphi = (\forall x)\psi$, was obtained from $T \vdash \psi$ by the inference rule FOR. Then:

$T \vdash \psi$ by the assumption
$K_iT \vdash K_i\psi$ by the induction hypothesis
$K_iT \vdash (\forall x)K_i\psi$ by FOR
$K_iT \vdash K_i(\forall x)\psi$ by Barcan formula.

(2) Suppose that $T \vdash \varphi$ where $\varphi = \Phi_{k,\theta,X}(C_G^r\psi)$ was derived by application of RPC. Then:

$T \vdash \Phi_{k,\theta,X}((F_G^r)^m\psi)$, for all $m \in \mathbb{N}$
$K_iT \vdash K_i\Phi_{k,\theta,X}((F_G^r)^m\psi)$, for all $m \in \mathbb{N}$, by induction hypothesis
$K_iT \vdash \top \to K_i\Phi_{k,\theta,X}((F_G^r)^m\psi)$, for all $m \in \mathbb{N}$

$K_iT \vdash \Phi_{k+1,\overline{\theta},\overline{X}}((F_G^r)^m\psi)$, where $\overline{\theta} = (\theta, \top)$ and $\overline{X} = (X, K_i)$.
$K_iT \vdash \Phi_{k+1,\overline{\theta},\overline{X}}(C_G^r\psi)$, by RPC
$K_iT \vdash \top \to K_i\Phi_{k,\theta,X}(C_G^r\psi)$
$K_iT \vdash \top \to K_i\varphi$
$K_iT \vdash K_i\varphi$. □

Remark 3 (k−nested Formulas in the Infinitary Rules). Note that in a finitary axiom system, the Strong necessitation theorem would be a direct consequence of the Necessitation inference rule and Distribution axiom, due to the fact that any proof uses a finite set of premises.

In the presence of the rules of the form (4.2), the derivations can be infinite and the proof that $T \vdash \psi$ implies $K_iT \vdash K_i\psi$, by induction on the length of the proof $T \vdash \psi$, must also include the case of application of Equation (4.2), assuming $T \vdash \psi_m$ for every $m \geq 0$. In the induction step, we use the hypothesis that $K_iT \vdash K_i\psi_m$, for every m. Then the presence of knowledge operator K_i prevents us to directly apply Equation (4.2) on the set $\{K_i\psi_m \mid m \geq 0\}$ to infer $K_iT \vdash K_i\psi$. For that reason, it is convenient to generalize the rule (4.2) by allowing all inferences of the form

$$\frac{\{K_{i_n}K_{i_{n-1}}\dots K_{i_1}\psi_m \mid m \geq 0\}}{K_{i_n}K_{i_{n-1}}\dots K_{i_1}\psi}, \tag{4.4}$$

for every integer n and every block of knowledge operators $K_{i_n}K_{i_{n-1}}\dots K_{i_1}$. With this modification, the case of application of Equation (4.4) in the inductive proof would be considered as follows: Let us denote $\overline{K} = K_{i_n}K_{i_{n-1}}\dots K_{i_1}$ and assume that $T \vdash \overline{K}\psi$ is obtained by Equation (4.4), so $T \vdash \overline{K}\psi_m$ for every m. Then, $K_iT \vdash K_i\overline{K}\psi_m$ for every m by the induction hypothesis, which allows us to obtain $K_iT \vdash K_i\overline{K}\psi$ by applying the instance $\frac{\{K_i\overline{K}\psi_m \mid m\geq 0\}}{K_i\overline{K}\psi}$ of Equation (4.4).

This idea is directly applied to the case (2) (rule RPC) in the previous proof, and it can be applied in the same way for the remaining three infinitary rules. Since the rules must formally be in an implicative form because of the Deduction theorem (see Remark 2), we used the implication with the antecedent $\top$.

In our approach, we need to meet both requirements of implicative forms and nesting of knowledge operators, i.e., our infinitary rules must generalize both Equations (4.3) and (4.4), which directly leads to our form of k-nested implication in Definition 2.3.

As a consequence, we also obtain strong necessitation for the operators of group knowledge. As we will see later, this result is necessary to prove so-called fixed-point axiom for common knowledge operator.

Corollary 4.3. *If T is a theory and $T \vdash \varphi$, then $E_GT \vdash E_G\varphi$, for all $G \subseteq \mathcal{A}$.*

Proof. Let $T \vdash \varphi$. For every $i \in G$, we have $E_GT \vdash K_iT$ by the axiom AE, and $K_iT \vdash K_i\varphi$, by Theorem 4.2. Since by the rule RE, where we choose $k = 0$ and $\theta_0 = \top$, we have

$$\{K_i\varphi \mid i \in G\} \vdash E_G\varphi,$$

and we obtain $E_GT \vdash E_G\varphi$. □

Now, we show that some standard properties of epistemic operators can be proved in $Ax_{PCK^{fo}}$.

Proposition 4.4. *Let $\varphi, \psi, \varphi_j, j = 1, \ldots, m$ be formulas, $i \in \mathcal{A}$ and $G \in \mathcal{G}$. Then:*

(1) $\vdash K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
(2) $\vdash E_G(\varphi \to \psi) \to (E_G\varphi \to E_G\psi)$
(3) $\vdash C_G(\varphi \to \psi) \to (C_G\varphi \to C_G\psi)$
(4) $\vdash K_i(\bigwedge_{j=1}^{m} \varphi_j) \equiv \bigwedge_{j=1}^{m} K_i\varphi_j, \forall i \in G,$
(5) $\vdash E_G(\bigwedge_{j=1}^{m} \varphi_j) \equiv \bigwedge_{j=1}^{m} E_G\varphi_j$
(6) $\vdash C_G\varphi \to E_G(\varphi \wedge C_G\varphi)$.

Proof.

(1) follows directly from AK.
(2) We use the following derivation:

$$\begin{aligned} E_G\varphi \wedge E_G(\varphi \to \psi) &\vdash \{K_i\varphi \wedge K_i(\varphi \to \psi) \mid \forall i \in G\} \text{ (by AE)} \\ &\vdash \{K_i\psi \mid \forall i \in G\} \text{ (by AK)} \\ &\vdash E_G\psi \text{ (by RE).} \end{aligned}$$

Therefore, by Deduction theorem $\vdash E_G\varphi \wedge E_G(\varphi \to \psi) \to E_G\psi$, i.e., $\vdash E_G(\varphi \to \psi) \to (E_G\varphi \to E_G\psi)$.
(3) Let us first prove, using the induction on n, that

$$\vdash (E_G)^m(\varphi \to \psi) \to ((E_G)^m\varphi \to (E_G)^m\psi) \tag{4.5}$$

holds for every $m \in \mathbb{N}$.
Induction base is proved in the previous part of this proposition (2).
Induction step:
$\vdash (E_G)^m(\varphi \to \psi) \to ((E_G)^m\varphi \to (E_G)^m\psi)$, induction hypothesis
$\vdash K_i((E_G)^m(\varphi \to \psi) \to ((E_G)^m\varphi \to (E_G)^m\psi)), \forall i \in G$, by RK
$\vdash E_G((E_G)^m(\varphi \to \psi) \to ((E_G)^m\varphi \to (E_G)^m\psi))$, by RE
$\vdash E_G((E_G)^m(\varphi \to \psi) \to ((E_G)^m\varphi \to (E_G)^m\psi)) \to (E_G^{m+1}(\varphi \to \psi) \to E_G((E_G)^m\varphi \to (E_G)^m\psi))$, by induction base
$\vdash (E_G)^{m+1}(\varphi \to \psi) \to E_G((E_G)^m\varphi \to (E_G)^m\psi)$, by previous two
$\vdash E_G((E_G)^m\varphi \to (E_G)^m\psi) \to ((E_G)^{m+1}\varphi \to (E_G)^{m+1}\psi)$, by induction base
$\vdash (E_G)^{m+1}(\varphi \to \psi) \to ((E_G)^{m+1}\varphi \to (E_G)^{m+1}\psi)$, by previous two.
Thus, Equation (4.5) holds. Next,

$$\begin{aligned} C_G\varphi \wedge C_G(\varphi \to \psi) &\vdash \{(E_G)^m\varphi \wedge (E_G)^m(\varphi \to \psi) \mid \forall m \in \mathbb{N}\} \text{ (by AC)} \\ &\vdash \{(E_G)^m\psi \mid \forall m \in \mathbb{N}\} \text{ (by (4.5))} \\ &\vdash C_G\psi \text{ (by RC).} \end{aligned}$$

Then, $\vdash C_G(\varphi \to \psi) \to (C_G\varphi \to C_G\psi)$, by the Deduction theorem.
(4) This standard result in modal logics follows from the Distribution axiom and propositional reasoning.

(5) First, we prove that $E_G(\bigwedge_{j=1}^{m} \varphi_j)$ implies $\bigwedge_{j=1}^{m} E_G\varphi_j$.

$$
\begin{aligned}
E_G\left(\bigwedge_{j=1}^{m} \varphi_j\right) &\vdash \left\{K_i\left(\bigwedge_{j=1}^{m} \varphi_j\right) \mid i \in G\right\} \text{ (by AE)}\\
&\vdash \left\{\bigwedge_{j=1}^{m} K_i\varphi_j \mid i \in G\right\} \text{ (by the previous part of the proposition (4))}\\
&\vdash \bigcup_{j=1}^{m}\{K_i\varphi_j \mid i \in G\}\left(\text{since } \bigwedge_{j=1}^{m} K_i\varphi_j \to K_i\varphi_j,\ \forall j = 1,\dots,m\right)\\
&\vdash \bigcup_{j=1}^{m}\{E_G\varphi_j \mid i \in G\} \text{ (by RE)}\\
&\vdash \bigwedge_{j=1}^{m} E_G\varphi_j \quad \text{(by propositional reasoning).}
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
\bigwedge_{j=1}^{m} E_G\varphi_j &\vdash \{K_i\varphi_1 \mid i \in G\} \cup \{K_i\varphi_2 \mid i \in G\} \cup \ldots \cup \{K_i\varphi_m \mid i \in G\} \text{ (by AE)}\\
&\vdash \left\{\bigwedge_{j=1}^{m} K_i\varphi_j \mid i \in G\right\}\\
&\vdash \left\{K_i\left(\bigwedge_{j=1}^{m} \varphi_j\right) \mid i \in G\right\} \text{ (by the previous part of the proposition (4))}\\
&\vdash E_G\left(\bigwedge_{j=1}^{m} \varphi_j\right) \quad \text{(by RE).}
\end{aligned}
$$

Therefore, by the Deduction theorem, we have that $\vdash E_G(\bigwedge_{j=1}^{m} \varphi_j) \equiv \bigwedge_{j=1}^{m} E_G\varphi_j$.

(6) $\vdash C_G\varphi \to E_G\{(E_G)^m\varphi \mid m \in \mathbb{N}\}$, by AC
$E_G\{(E_G)^m\varphi \mid m \in \mathbb{N}\} \vdash E_GC_G\varphi$, by RC and Corollary 4.3
$\vdash C_G\varphi \to E_GC_G\varphi$, by previous two
$\vdash C_G\varphi \to E_G\varphi$, by AC
$\vdash C_G\varphi \to E_G(\varphi \wedge C_G\varphi)$, by previous two and the previous part (5) of the proposition. □

Note that (3) and (6) (the fixed-point axiom) are two standard axioms of epistemic logic with common knowledge [17, 24]. The axiom (3) is often written in an equivalent form

$$(C_G\varphi \wedge C_G(\varphi \to \psi)) \to C_G\psi.$$

The previous result shows that they are provable in our axiomatic system $Ax_{PCK^{fo}}$.

The standard axiomatization for epistemic logics (with finitely many agents) [17, 24] also includes one axiom for group knowledge operator, which states that group knowledge $E_G\varphi$ is equivalent to the conjunction of $K_i\varphi$, where all the agents i from the group are considered. The next result shows that both that axiom and its probabilistic variant hold in our logic.

Proposition 4.5. *Let φ be a formula, $r \in [0,1]_{\mathbb{Q}}$, and let $G \in \mathcal{G}$ be a finite set of agents. Then the following hold:*

(1) $\vdash E_G\varphi \equiv \bigwedge_{i\in G} K_i\varphi$
(2) $\vdash E^r_G\varphi \equiv \bigwedge_{i\in G} K^r_i\varphi$.

Proof.

(1) From the axiom AE, using propositional reasoning, we can obtain $\vdash E_G\varphi \rightarrow \bigwedge_{i\in G} K_i\varphi$. However, from the inference rule RE, choosing $k = 0$ and $\theta_0 = \top$, we obtain $\{K_i\varphi \mid i \in G\} \vdash E_G\varphi$, i.e., $\bigwedge_{i\in G} K_i\varphi \vdash E_G\varphi$, so $\vdash \bigwedge_{i\in G} K_i\varphi \rightarrow E_G\varphi$ follows from the Deduction theorem.
(2) This result can be proved in the same way as the first statement, using the obvious analogies between the axioms AE and APE, and the rules RE and RPE. □

Note that the distribution properties of the epistemic operators K_i, E_G, and C_G, proved in Proposition 4.4 (1)–(3), cannot be directly transferred to the properties of the corresponding operators of probabilistic knowledge. For example, it is easy to see that $E^r_G(\varphi \rightarrow \psi) \rightarrow (E^r_G\varphi \rightarrow E^r_G\psi)$ is not a valid formula.[5] Nevertheless, we can prove that probabilistic versions of knowledge, group knowledge, and common knowledge are closed under consequences.

Proposition 4.6. *Let φ and ψ be formulas such that $\vdash \varphi \rightarrow \psi$. Let $r \in [0,1]_{\mathbb{Q}}$, $i \in \mathcal{A}$, and $G \in \mathcal{G}$. Then:*

(1) $\vdash K^r_i\varphi \rightarrow K^r_i\psi$
(2) $\vdash E^r_G\varphi \rightarrow E^r_G\psi$
(3) $\vdash C^r_G\varphi \rightarrow C^r_G\psi$.

Proof.

(1) Note that

$$\vdash K_i(P_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow (P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi)) \rightarrow (K_iP_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow K_i(P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi)) \tag{4.6}$$

by Proposition 4.4(1). From the assumption $\vdash \varphi \rightarrow \psi$, applying the rule RP and then the rule RK, we obtain

$$\vdash K^1_i(\varphi \rightarrow \psi). \tag{4.7}$$

Note that $\vdash \neg\varphi \vee \neg\bot$ (a propositional tautology), so

$$\vdash P_{i,\geq 1}(\neg\varphi \vee \neg\bot), \text{ by RP.} \tag{4.8}$$

Also, $\vdash \neg(\varphi \wedge \neg\bot) \vee \neg\neg\varphi$, so

$$\vdash P_{i,\geq 1}(\neg(\varphi \wedge \neg\bot) \vee \neg\neg\varphi), \text{ by RP.} \tag{4.9}$$

By P4, we have $\vdash (P_{i,\geq r}\varphi \wedge P_{i,\geq 0}\neg\bot \wedge P_{i,\geq 1}(\neg\varphi \vee \neg\bot)) \rightarrow P_{i,\geq 1}(\varphi \vee \bot)$, so

$$\vdash P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}(\varphi \vee \bot), \text{ by (4.8) using the instance } P_{i,\geq 0}\neg\bot \text{ of P1.} \tag{4.10}$$

The formula $P_{i,\geq r}(\varphi \vee \bot)$ denotes $P_{i,\geq r}\neg(\neg\varphi \wedge \neg\bot)$, which is the same as $P_{i,\geq 1-(1-r)}\neg(\neg\varphi \wedge \neg\bot)$, and can be abbreviated as $P_{i,\leq 1-r}(\neg\varphi \wedge \neg\bot)$. Similarly, $\neg P_{i,\geq r}\neg\neg\varphi$ denotes $P_{i,<r}\neg\neg\varphi$. From P5, we obtain $\vdash (P_{i,\leq 1-r}(\neg\varphi \wedge \neg\bot) \wedge P_{i,<r}\neg\neg\varphi) \rightarrow P_{i,<1}((\neg\varphi \wedge \neg\bot) \vee \neg\neg\varphi)$.

[5] However, it can be shown that the formula $E^1_G(\varphi \rightarrow \psi) \rightarrow (E^r_G\varphi \rightarrow E^r_G\psi)$ is valid and it is a theorem of our logic (see (4.17)).

Since $P_{i,\geq 1}(\neg(\varphi \wedge \neg\bot) \vee \neg\neg\varphi)$ denotes $\neg P_{i,<1}((\neg\varphi \wedge \neg\bot) \vee \neg\neg\varphi)$, from (4.9), we have $\vdash (P_{i,\leq 1-r}(\neg\varphi \wedge \neg\bot) \wedge P_{i,<r}\neg\neg\varphi) \rightarrow P_{i,<1}((\neg\varphi \wedge \neg\bot) \vee \neg\neg\varphi) \wedge \neg P_{i,<1}((\neg\varphi \wedge \neg\bot) \vee \neg\neg\varphi))$, by P5, and therefore $\vdash P_{i,\leq 1-r}(\neg\varphi \wedge \neg\bot) \rightarrow P_{i,<r}\neg\neg\varphi$, i.e.,

$$\vdash P_{i,\geq r}(\varphi \vee \bot) \rightarrow P_{i,\geq r}\neg\neg\varphi. \tag{4.11}$$

From (4.10) and (4.11), we obtain $\vdash P_{i,\geq r}(\varphi) \rightarrow P_{i,\geq r}\neg\neg\varphi$. The negation of the formula

$$P_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow (P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi) \tag{4.12}$$

is equivalent to $P_{i,\geq 1}(\neg\varphi \vee \psi) \wedge P_{i,\geq r}\varphi \wedge P_{i,<r}\psi$. Since $P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\neg\neg\varphi$, then $P_{i,\geq 1}(\neg\varphi \vee \psi) \wedge P_{i,\geq r}\neg\neg\varphi \wedge P_{i,<r}\psi$, which can be written as $P_{i,\geq 1}(\neg\varphi \vee \psi) \wedge P_{i,\leq 1-r}\neg\varphi \wedge P_{i,<r}\psi$. Then, $\vdash P_{i,\leq 1-r}\neg\varphi \wedge P_{i,<r}\psi \rightarrow P_{i,<r}(\neg\varphi \vee \psi)$, by P5, and, since $P_{i,<1}\varphi$ is an abbreviation for $\neg P_{i,\geq 1}\varphi$, we have $\vdash \neg(P_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow (P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi)) \rightarrow P_{i,\geq 1}(\neg\varphi \vee \psi) \wedge \neg P_{i,\geq 1}(\neg\varphi \vee \psi)$, a contradiction. Thus, the formula (4.12) is a theorem of our axiomatization. By applying the rule RK to the theorem, we obtain

$$\vdash K_i(P_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow (P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi)). \tag{4.13}$$

From (4.6) and (4.13), we obtain

$$\vdash K_iP_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow K_i(P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi). \tag{4.14}$$

By Proposition 4.4(1), we have

$$\vdash K_i(P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi) \rightarrow (K_iP_{i,\geq r}\varphi \rightarrow K_iP_{i,\geq r}\psi). \tag{4.15}$$

From (4.14) and (4.15), we obtain $\vdash K_iP_{i,\geq 1}(\varphi \rightarrow \psi) \rightarrow (K_iP_{i,\geq r}\varphi \rightarrow K_iP_{i,\geq r}\psi)$, i.e.,

$$\vdash K_i^1(\varphi \rightarrow \psi) \rightarrow (K_i^r\varphi \rightarrow K_i^r\psi). \tag{4.16}$$

Finally, from (4.7) and (4.16), we obtain $\vdash K_i^r\varphi \rightarrow K_i^r\psi$.

(2) We start with the following derivation:

$$\begin{aligned} E_G^r\varphi \wedge E_G^1(\varphi \rightarrow \psi) &\vdash \{K_i^r\varphi \wedge K_i^1(\varphi \rightarrow \psi) \mid \forall i \in G\}, \text{ by APE} \\ &\vdash \{K_i^r\psi \mid \forall i \in G\}, \text{ by (4.16)} \\ &\vdash E_G^r\psi, \text{ by RPE.} \end{aligned}$$

Therefore,

$$\vdash E_G^1(\varphi \rightarrow \psi) \rightarrow (E_G^r\varphi \rightarrow E_G^r\psi) \tag{4.17}$$

by the Deduction theorem. From (4.8), using the rule RPE, we obtain

$$\vdash E_G^1(\varphi \rightarrow \psi). \tag{4.18}$$

Finally, from (4.17) and (4.18), we obtain $\vdash E_G^r\varphi \rightarrow E_G^r\psi$.

(3) First, we prove that

$$\vdash (F_G^r)^m\varphi \rightarrow (F_G^r)^m\psi \tag{4.19}$$

holds for every m. We prove the claim by induction.

Induction base follows trivially, since $(F_G^r)^0\varphi = \top$.

Suppose that $\vdash (F_G^r)^m\varphi \rightarrow (F_G^r)^m\psi$ (induction hypothesis).

$\vdash (\varphi \wedge (F_G^r)^m\varphi) \rightarrow (\varphi \wedge (F_G^r)^m\psi)$

$\vdash P_{i,\geq 1}((\varphi \wedge (F_G^r)^m\varphi) \rightarrow (\varphi \wedge (F_G^r)^m\psi)), \forall i \in G$, by RP

$\vdash K_iP_{i,\geq 1}((\varphi \wedge (F_G^r)^m\varphi) \rightarrow (\varphi \wedge (F_G^r)^m\psi)), \forall i \in G$, by RK

$\vdash E_G^1((\varphi \wedge (F_G^r)^m\varphi) \rightarrow (\varphi \wedge (F_G^r)^m\psi))$, by RPE

$\vdash E_G^1((\varphi \wedge (F_G^r)^m\varphi) \rightarrow (\varphi \wedge (F_G^r)^m\psi)) \rightarrow (E_G^r(\varphi \wedge (F_G^r)^m\varphi) \rightarrow E_G^r(\varphi \wedge (F_G^r)^m\psi))$, by (4.17)

$\vdash E_G^r(\varphi \wedge (F_G^r)^m\varphi) \rightarrow E_G^r(\varphi \wedge (F_G^r)^m\psi)$ by previous two, i.e.,

$\vdash (F_G^r)^{m+1}\varphi \to (F_G^r)^{m+1}\psi$.
Thus, (4.19) holds.

$$\begin{aligned} C_G^r\varphi &\vdash \{(F_G^r)^m\varphi \mid \forall m \in \mathbb{N}_0\} \text{ (by APC)} \\ &\vdash \{(F_G^r)^m\psi \mid \forall m \in \mathbb{N}_0\}, \text{ by (4.19)} \\ &\vdash C_G^r\psi, \text{ by RPC.} \end{aligned}$$

Now $\vdash C_G^r\varphi \to C_G^r\psi$ follows from the Deduction theorem. □

Next, we prove several results about maximal consistent sets with respect to our axiomatic system. Those results will be useful in proving the Truth lemma.

Lemma 4.7. *Let T be a maximal consistent set of formulas for $Ax_{PCK^{fo}}$. Then, T satisfies the following properties:*

(1) for every formula φ, exactly one of φ and $\neg\varphi$ is in T,
(2) T is deductively closed,
(3) $\varphi \wedge \psi \in T$ iff $\varphi \in T$ and $\psi \in T$,
(4) if $\{\varphi, \varphi \to \psi\} \subseteq T$, then $\psi \in T$,
(5) if $r = \sup\{q \in [0,1]_{\mathbb{Q}} \mid P_{i,\geq q}\varphi \in T\}$ and $r \in [0,1]_{\mathbb{Q}}$, then $P_{i,\geq r}\varphi \in T$.

Proof.

(1) If both formulas $\varphi, \neg\varphi \in T$, then T would be inconsistent. Suppose $\varphi \notin T$. Since T is maximal, $T \cup \{\varphi\}$ is inconsistent, and by the Deduction theorem $T \vdash \neg\varphi$. Similarly, if $\neg\varphi \notin T$, then $T \vdash \varphi$. Therefore, if both formulas $\varphi, \neg\varphi \notin T$, set T would be inconsistent, so exactly one of them is in T.
(2) Otherwise, if there is some φ such that $T \vdash \varphi$ and $\varphi \notin T$, then, by the previous part of this lemma, $\neg\varphi \in T$, so T would be inconsistent.
(3) Suppose $\varphi \in T$ and $\psi \in T$. Then, $T \vdash \varphi$, $T \vdash \psi$, $T \vdash \varphi \wedge \psi$ and $\varphi \wedge \psi \in T$, because T is deductively closed by Lemma 4.7(2). For the other direction, let $\varphi \wedge \psi \in T$. Then, $T \vdash \varphi \wedge \psi$, $T \vdash (\varphi \wedge \psi) \to \varphi$, $T \vdash (\varphi \wedge \psi) \to \psi$, $T \vdash \varphi$ and $T \vdash \psi$. Therefore, $\varphi, \psi \in T$, by Lemma 4.7(2).
(4) If $\{\varphi, \varphi \to \psi\} \subseteq T$, then $T \vdash \varphi$, $T \vdash \varphi \to \psi$ and $T \vdash \psi$, so $\psi \in T$ by Lemma 4.7(2).
(5) Let $r = \sup\{q \mid P_{i,\geq q}\varphi \in T\}$, thus $T \vdash P_{i,\geq q}\varphi$ for every $q < r$, $q \in [0,1]_{\mathbb{Q}}$. Then, by the Archimedean rule RA, we have that $T \vdash P_{i,\geq r}\varphi$. Therefore, $P_{i,\geq r}\varphi \in T$ by Lemma 4.7(2). □

Lemma 4.8. *Let V be a maximal consistent set of formulas.*

(1) $E_G\varphi \in V$ iff ($K_i\varphi \in V$ for all $i \in G$)
(2) $E_G^r\varphi \in V$ iff ($K_i^r\varphi \in V$ for all $i \in G$)
(3) $C_G\varphi \in V$ iff ($(E_G)^m\varphi \in V$ for all $m \in \mathbb{N}$)
(4) $C_G^r\varphi \in V$ iff ($(F_G^r)^m\varphi \in V$ for all $m \in \mathbb{N}$),

Proof. For the proof of (1), suppose that $E_G\varphi \in V$. Since $E_G\varphi \to K_i\varphi$, for all $i \in G$ is the axiom AE, then also $E_G\varphi \to K_i\varphi \in V$ for all $i \in G$. Therefore, $K_i\varphi \in V$ for all $i \in G$ by Lemma 4.7(4), because V is maximal consistent. For the other direction, if $K_i\varphi \in V$ for all $i \in G$, and, since $\{K_i\varphi \mid i \in G\} \vdash E_G\varphi$ (by the rule RE, where $k = 0$ and $\theta_0 = \top$), we have that $E_G\varphi \in V$, by Lemma 4.7(2).

The cases (2), (3), and (4) can be proved in a similar way, by replacing $E_G\varphi$, $K_i\varphi$, for all $i \in G$, axiom AE, and rule RE with $E_G^r\varphi$, $K_i^r\varphi$, for all $i \in G$, APE, RPE (case (2)), $C_G\varphi$, $(E_G)^m\varphi$, for all $m \in \mathbb{N}$, AC, RC (case (3)), and $C_G^r\varphi$, $(F_G^r)^m\varphi$ for all $m \in \mathbb{N}$, APC, RPC (case (4)), respectively. □

5 COMPLETENESS

In this section, we prove that the axiomatic system $Ax_{PCK^{fo}}$ is strongly complete with respect to the class of measurable $\mathcal{M}_{\mathcal{A}}^{MEAS}$ models. We will use Henkin-style construction [28], extending our language with infinitely many new constant symbols, known as witnessing constants, in a particular manner. We prove completeness in three main steps:

- First, we prove Lindenbaum's theorem, i.e., we show how to extend a consistent set of sentences T to a saturated maximal consistent set of sentences T^* step-by-step, in an infinite process, considering in each step one sentence and checking its consistency with the considered theory in that step. Due to the presence of infinitary rules, we modify the standard completion technique in the case that the considered sentence can be derived by an infinitary rule, by adding the negation of one of the premises of the rule. Such a premise always exists, for every considered infinitary rule, due to the presence of the Deduction theorem. We will discuss further necessity of this modification of the standard completion technique in Remark 4.
- Second, we use saturated sets to construct a special PCK^{fo} model M^*, which we will call the *canonical model*, and we show that it belongs to the class $\mathcal{M}_{\mathcal{A}}^{MEAS}$. The states of this model correspond to saturated theories. The key step in proving that M^* is a measurable model is the Truth lemma (Lemma 5.4), which ensures that a formula belongs to a saturated theory iff it is satisfied in the corresponding state of the canonical model. The proof of the lemma is by induction on complexity of the formula, and the important case when a formula is obtained by application of a knowledge operator essentially depends on the Strong necessitation theorem.
- Finally, using the saturation T^* of the considered theory T, we show that T is satisfiable in the corresponding state s_{T^*} of the canonical model.

5.1 Lindenbaum's Theorem

We start with the Henkin construction of saturated extensions of theories. For that purpose, we consider a broader language, obtained by adding countably many novel constant symbols. First, we motivate our modification of the standard proof strategy.

Remark 4. The standard process of extending a theory to a maximal consistent theory assumes an enumeration $\{\varphi_i \mid i \in \mathbb{N}\}$ of all sentences, and in the step k of the process the formula φ_k is considered and added to the current theory in the case that it is consistent with it. This standard infinite process is not directly applicable if the axiomatization contains infinitary rules of inference, as the resulting theory might be inconsistent even if in each step we ensure consistency. To illustrate that fact, let us consider the theory $T_0 = \{\neg C_G^r \varphi\}$, for some sentence φ, and the sequence of formulas $\varphi_1, \varphi_2, \varphi_3, \ldots$, where $\varphi_i = (F_G^r)^i \varphi$ for every i. Then, at each step k, the set $T_k = T_{k-1} \cup \{\varphi_k\}$ is consistent, but the set $\bigcup_{i=0}^{+\infty} T_i$ is not, since $\{\varphi_1, \varphi_2, \varphi_3, \ldots\} \vdash C_G^r \varphi$, by RPC.

For that reason, we modify the procedure by considering an enumeration of all sentences and adding, in the case when the considered sentence can be derived by an infinitary rule, the negation of one of the premises of the rule. We will see in the proof of the following result that such a premise always exists, as a consequence of the Deduction theorem.

Theorem 5.1 (Lindenbaum's Theorem). *Let T be a consistent theory in the language $\mathcal{L}_{PCK^{fo}}$, and C an infinite enumerable set of new constant symbols (i.e., $C \cap \mathcal{L}_{PCK^{fo}} = \emptyset$). Then, T can be extended to a saturated theory T^* in the language $\mathcal{L}^* = L_{PCK^{fo}} \cup C$.*

Proof. Let $\{\varphi_i \mid i \in \mathbb{N}\}$ be an enumeration of all sentences in $Sent_{PCK^{fo}}$. Let C be an infinite enumerable set of constant symbols such that $C \cap \mathcal{L}_{PCK^{fo}} = \emptyset$. We define the family of theories $(T_i)_{i \in \mathbb{N}}$ and the set T^* in the following way:

(1) $T_0 = T$.
(2) For every $i \in \mathbb{N}$:
   (a) if $T_i \cup \{\varphi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\varphi_i\}$
   (b) if $T_i \cup \{\varphi_i\}$ is inconsistent, and
      (b1) $\varphi_i = \Phi_{k,\theta,X}(E_G\varphi)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\Phi_{k,\theta,X}(K_j\varphi)\}$, for some $j \in G$ such that T_{i+1} is consistent;
      (b2) $\varphi_i = \Phi_{k,\theta,X}(C_G\varphi)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\Phi_{k,\theta,X}((E_G)^m\varphi)\}$, for some $m \in \mathbb{N}$ such that T_{i+1} is consistent;
      (b3) $\varphi_i = \Phi_{k,\theta,X}(E^r_G\varphi)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\Phi_{k,\theta,X}(K^r_j\varphi)\}$, for some $j \in G$ such that T_{i+1} is consistent;
      (b4) $\varphi_i = \Phi_{k,\theta,X}(C^r_G\varphi)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\Phi_{k,\theta,X}((F^r_G)^m\varphi)\}$, for some $m \in \mathbb{N}$ such that T_{i+1} is consistent;
      (b5) $\varphi_i = \Phi_{k,\theta,X}(P_{i,\geq r}\varphi)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\Phi_{k,\theta,X}(P_{i,\geq r-\frac{1}{m}}\varphi)\}$, for some $m \in \mathbb{N}$ such that T_{i+1} is consistent;
      (b6) $\varphi_i = (\forall x)\varphi(x)$, then $T_{i+1} = T_i \cup \{\neg\varphi_i, \neg\varphi(c)\}$, for some constant symbol $c \in C$, which does not occur in any of the formulas from T_i such that T_{i+1} remains consistent
   (c) Otherwise, $T_{i+1} = T_i \cup \{\neg\varphi_i\}$.
(3) $T^* = \bigcup_{i=0}^{\infty} T_i$.

First, we need to prove that the set T^* is well defined, i.e., we need to show that the agents $j \in G$ used the steps (b1) and (b3) exist, that the numbers $m \in \mathbb{N}$ used in the steps (b2), (b4), and (b5) exist, and that the constant $c \in C$ from step (b6) exists. Let us prove correctness in step (b4) exists, i.e., that if $T_i \cup \{\Phi_{k,\theta,X}(C^r_G\varphi)\}$ is inconsistent, then there exists $m \geq 1$ such that $T_i \cup \{\neg\Phi_{k,\theta,X}((F^r_G)^m\varphi)\}$ is consistent. Otherwise, if $T_i \cup \{\neg\Phi_{k,\theta,X}((F^b_G)^m\varphi)\}$ would be inconsistent for every m, then $T_i \vdash \Phi_{k,\theta,X}((F^r_G)^m\varphi)$ for each m by the Deduction theorem, and therefore $T_i \vdash \Phi_{k,\theta,X}(C^r_G\varphi)$ by the inference rule RPC. But, since $T_i \cup \{\Phi_{k,\theta,X}(C^r_G\varphi)\}$ is inconsistent, we have $T_i \vdash \neg\Phi_{k,\theta,X}(C^r_G\varphi)$, which is in a contradiction with consistency of T_i. In a similar way, we can prove existence of j and m in steps (b1–b5), where the other infinitary rules are considered. Let us now consider the case (b6). It is obvious that the formula $\neg(\forall x)\varphi(x)$ can be consistently added to T_i, and if there is already some $c \in C$ such that $\neg\varphi(c) \in T_i$, the proof is finished. If there is no such c, then observe that T_i is constructed by adding finitely many formulas to T, so there is a constant symbol $c \in C$ that does not appear in T_i. Let us show that we can choose that c in (b6). If we suppose that $T_i \cup \{\neg(\forall x)\varphi(x), \neg\beta(c)\} \vdash \bot$, then by the Deduction theorem, we have $T_i, \neg(\forall x)\varphi(x) \vdash \varphi(c)$. Note that c does not appear in $T_i \cup \{\neg(\forall x)\varphi(x)\}$, and therefore, $T_i, \neg(\forall x)\varphi(x) \vdash (\forall x)\varphi(x)$, which is impossible. Thus, the sets T_i are well defined. Note that they are consistent by construction.

Next, we prove that T^* is deductively closed, using the induction on the length of proof. The proof is straightforward in the case of finitary rules. Here, we will only prove that T^* is closed under the rule RPC, since the cases when other infinitary rules are considered can be treated in a similar way.

Suppose $T^* \vdash \phi$ was obtained by RPC, where $\Phi_{k,\theta,X}((F^r_G)^n\varphi) \in T^*$ for all $n \in \mathbb{N}$, and $\phi = \Phi_{k,\theta,X}(C^r_G\varphi)$. Assume that $\Phi_{k,\theta,X}(C^r_G\varphi) \notin T^*$. Let i be the positive integer such that $\varphi_i = \Phi_{k,\theta,X}(C^r_G\varphi)$. Then, $T_i \cup \{\varphi_i\}$ is inconsistent, since otherwise $\Phi_{k,\theta,X}(C^r_G\varphi) = \varphi_i \in T_{i+1} \subset T^*$.

Therefore, $T_{i+1} = T_i \cup \{\neg\Phi_{k,\theta,X}((F_G^r)^m\varphi)\}$ for some m, so $\neg\Phi_{k,\theta,X}((F_G^r)^m\varphi) \in T^*$, which contradicts the consistency of T_j.

If we would suppose T is inconsistent, i.e., $T^* \vdash \bot$, then we would have $\bot \in T^*$, since T^* is deductively closed. Therefore, there would be some i such that $\bot \in T_i$, which is impossible. Thus, T^* is consistent.

Finally, the step (b6) of the construction guarantees that the theory T^* is saturated in the extended language $\mathcal{L}^*$. □

5.2 Canonical Model

Now, we construct a special Kripke structure whose set of states consists of saturated theories. First, we need to introduce some notation. For a given set of formulas T and $i \in \mathcal{A}$, we define the set T/K_i as the set of all formulas φ, such that $K_i\varphi$ belongs to T, i.e.,

$$T/K_i = \{\varphi \mid K_i\varphi \in T\}.$$

Definition 5.2 (Canonical Model). The *canonical model* is the structure $M^* = (S, D, I, \mathcal{K}, \mathcal{P})$, such that

- $S = \{s_V \mid V \text{ is a saturated theory}\}$;
- D is the set of all variable-free terms;
- $\mathcal{K}_i = \{(s_V, s_U) \mid V/K_i \subseteq U\}$, $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathcal{A}\}$;
- $I(s)$ is an interpretation such that:
  - for each function symbol f_j^k, $I(s)(f_j^k)$ is a function from D^k to D such that for all variable-free terms $t_1, \ldots, t_k$, $I(s)(f_j^k) :(t_1, \ldots, t_k) \to f_j^k(t_1, \ldots, t_k)$,
  - for each relational symbol R_j^k, $I(s)(R_j^k) = \{(t_1, \ldots, t_k) \mid t_1, \ldots, t_k$ are variable-free terms in $R_j^k(t_1, \ldots, t_k) \in V$, where $s = s_V\}$;
- $\mathcal{P}(i, s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$, where
  - $S_{i,s} = S$,
  - $\chi_{i,s} = \{[\varphi]_{i,s} \mid \varphi \in Sent_{PCK}\}$, where $[\varphi]_{i,s} = \{s_V \in S_{i,s} \mid \varphi \in V\}$,
  - if $[\varphi]_{i,s} \in \chi_{i,s}$, then $\mu_{i,s}([\varphi]_{i,s}) = \sup\{r \mid P_{i,\geq r}\varphi \in V, \text{ where } s = s_V\}$.

Note that the sets $[\varphi]_{i,s}$ in the definition of the canonical mode actually do not depend on i and s, so in the rest of this section, we will sometimes relax the notation by omitting the subscript.

Also, since there is a bijection between saturated theories and states of the canonical model, we will often write just s when we denote either state of the corresponding saturated theory. For example, we can write the last item of the definition above as $\mu_{i,s}([\varphi]_{i,s}) = \sup\{r \mid P_{i,\geq r}\varphi \in s\}$.

Now, we will show that M^* is a well-defined PCK^{fo} model, i.e., that:

- The definition of $\mu_{i,s}$ is correct, i.e., $\mu_{i,s}([\varphi]_{i,s})$ does not depend on the way we choose a sentence from the class $[\varphi]_{i,s}$;
- Each $\mathcal{P}(i, s)$ is a probability space, i.e., each $\chi_{i,s}$ is an algebra of subsets of $S_{i,s}$, and each $\mu_{i,s}$ is a finitely additive probability measure; and
- M^* is a measurable model. Note that above in the algebras $\chi_{i,s}$, sets $[\varphi]_{i,s}$ are defined using $\varphi \in s$, and not $(M^*, s) \models \varphi$, as it is required for PCK^{fo} models. We prove in Lemma 5.4 that those sets actually coincide.

First, we show that $\mathcal{P}(i, s)$ is a well-defined probability space.

Lemma 5.3. *Let $M^* = (S, D, I, \mathcal{K}, \mathcal{P})$ be the canonical model. Then, for each agent $i \in \mathcal{A}$ and $s \in S$, the following hold*[6]*:*

(1) If φ and ψ are two sentences such that $[\varphi]_{i,s} = [\psi]_{i,s}$, then $\sup\{r \mid P_{i,\geq r}\psi \in s\} = \sup\{r \mid P_{i,\geq r}\varphi \in s\}$
(2) $\mathcal{P}(i, s)$ is a is a probability space.

Proof.

(1) If $[\varphi]_{i,s} = [\psi]_{i,s}$, then φ and ψ belong to the same saturated theories, so $\vdash \varphi \equiv \psi$. From $\vdash \varphi \rightarrow \psi$, we obtain $\vdash P_{i,\geq 1}(\varphi \rightarrow \psi)$ by RA, and therefore for every r, we have $\vdash P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi$ by (4.12). Consequently, $P_{i,\geq r}\varphi \rightarrow P_{i,\geq r}\psi \in s$. If $P_{i,\geq r}\varphi \in s$, then, by Lemma 4.7(4), also $P_{i,\geq r}\psi \in s$. Therefore, $\sup\{r \mid P_{i,\geq r}\psi \in s\} \geq \sup\{r \mid P_{i,\geq r}\varphi \in s\}$. In the same way, we can prove $\sup\{r \mid P_{i,\geq r}\psi \in s\} \leq \sup\{r \mid P_{i,\geq r}\varphi \in s\}$ using $\vdash \psi \rightarrow \varphi$.

(2) First, we show that for each agent $i \in \mathcal{A}$ and $s \in S$, the class $\chi_{i,s} = \{[\varphi] \mid \varphi \in Sent_{PCK_\infty}\}$ is an algebra of subsets of $S_{i,s}$. Obviously, we have that $S_{i,s} = [\varphi \vee \neg\varphi]$, for every formula φ. Also, if $[\varphi] \in \chi_{i,s}$, then $[\neg\varphi]$ is a complement of the set $[\varphi]$, and it belongs to $\chi_{i,s}$ Finally, if $[\varphi_1], [\varphi_2] \in \chi_{i,s}$, then $[\varphi_1] \cup [\varphi_2] \in \chi_{i,s}$, because $[\varphi_1] \cup [\varphi_2] = [\varphi_1 \vee \varphi_2]$. Therefore, each $\chi_{i,s}$ is an algebra of subsets of $S_{i,s}$.

Note that from the axiom $P_{i,\geq o}\varphi$, we can obtain $\mu_{i,s}([\varphi]) \geq 0$. Next, we show $\mu_{i,s}([\varphi]) = 1 - \mu_{i,s}([\neg\varphi])$. Suppose $q = \mu_{i,s}([\varphi]) = sup\{r \mid P_{i,\geq r}\varphi \in s\}$. If $q = 1$, then $P_{i,\geq r}\varphi = P_{i,\leq 0}\neg\varphi = \neg P_{i,>0}\neg\varphi$ and $\neg P_{i,>0}\neg\varphi \in s$. If for some $l > 0$, $P_{i,\geq l}\neg\varphi \in s$, then $P_{i,>0}\neg\varphi \in s$, by axiom P2, which is a contradiction. Therefore, $\mu_{i,s}([\varphi]) = 1$. Suppose $q < 1$. Then, for every rational number $q' \in (q, 1]$, $\neg P_{i,\geq q'}\varphi = P_{i,<q'}\varphi$, so $P_{i,<q'}\varphi \in s$. Then, by P2, $P_{i,\leq q'}\varphi$ and $P_{i,\geq 1-q'}\neg\varphi \in s$. However, if there is a rational $q'' \in [0, r)$ such that $P_{i,\geq 1-q''}\neg\varphi \in s$, then $\neg P_{i,>q''} \in s$, which is a contradiction. Therefore, $sup\{r \mid P_{i,\geq r}\neg\varphi \in s\} = 1 - sup\{r \mid P_{i,\geq r}\varphi \in s\}$. Thus, $\mu_{i,s}([\varphi]) = 1 - \mu_{i,s}([\neg\varphi])$. Let $[\varphi]_{i,s} \cap [\psi]_{i,s} = \emptyset$, $\mu_{i,s}([\varphi]) = q$, $\mu_{i,s}([\psi]) = l$. Since $[\psi]_{i,s} \subset [\neg\varphi]_{i,s}$, it follows that $q + l \leq q + (1 - q) = 1$. Suppose that $q, l > 0$. Because of supremum and monotonicity properties, for all rational numbers $q' \in [0, q)$ and $l' \in [0, l)$: $P_{i,\geq q'}\varphi, P_{i,\geq l'}\psi \in s$. Then, $P_{i,\geq q'+l'}(\varphi \vee \psi) \in s$ by P4. Therefore, $q + l \leq sup\{r \mid P_{i,\geq r}(\varphi \vee \psi) \in s\}$. If $q + l = 1$, then the statement is obviously valid. Suppose $q + l < 1$. If $q + l < r_0 =\leq sup\{r \mid P_{i,\geq r}(\varphi \vee \psi) \in s\}$, then for each rational $r' \in (q + l, r_0)$, $P_{i,\geq r'}(\varphi \vee \psi) \in s$. Let us choose rational $q'' > q$ and $s'' > s$ such that $\neg P_{i,\geq q''}\varphi, P_{i,<q''}\varphi \in s$, $\neg P_{i,\geq l''}\psi, P_{i,<l''}\psi \in s$ and $q'' + l'' = r' \leq 1$. Then, $P_{i,\leq q''}\varphi \in s$ by the axiom P3. And by P5, we have $P_{i,\leq q''+'l''}(\varphi \vee \psi)$, $\neg P_{i,\geq q''+'l''}(\varphi \vee \psi)$ and $\neg P_{i,\geq r'}(\varphi \vee \psi)$, which is a contradiction. Therefore, $\mu_{i,s}([\varphi] \cup [\psi]) = \mu_{i,s}([\varphi]) + \mu_{i,s}([\psi])$. Finally, let us assume that $q = 0$ or $l = 0$. In that case, we can repeat the previous reasoning by taking either $q' = 0$ or $l' = 0$. □

The previous result still does not ensure that M^* belongs to the class $\mathcal{M}_{\mathcal{A}}^{MEAS}$. Indeed, in Definition 5.2, the sets $[\varphi]$ from $\chi_{i,s}$ are defined using $\varphi \in T$, and not $(M^*, s_T) \models \varphi$. However, the following lemma shows that the former and latter coincide:

Lemma 5.4 (Truth Lemma). *Let T be a saturated theory. Then,*

$$\varphi \in T \quad \textit{iff} \quad (M^*, s_T) \models \varphi.$$

[6]The proof of Lemma 5.3 is essentially the same as the proofs of corresponding statements in single-agent probability logics [41]. We present it here for the completeness of the article, and also because some steps in the proof will be useful for the proof of Theorem 6.3.

Proof. We prove the equivalence by induction on complexity of φ:

- If the formula φ is atomic, then $\varphi \in T$ iff $(M^*, s_T) \models \varphi$, by the definition of $I(s)$ in M^*;
- Let $\varphi = \neg\psi$. Then, $(M^*, s_T) \models \neg\psi$ iff $(M^*, s_T) \not\models \psi$ iff $\psi \notin T$ (induction hypothesis) iff $\neg\psi \in T$;
- Let $\varphi = \psi \wedge \eta$. Then, $(M^*, s_T) \models \psi \wedge \eta$ iff $(M^*, s_T) \models \psi$ and $(M^*, s_T) \models \eta$ iff $\psi \in T$ and $\eta \in T$ (induction hypothesis) iff $\psi \wedge \eta \in T$ by Lemma 4.7(3);
- Let $\varphi = (\forall x)\psi$ and $\varphi \in T$. Then, $\psi(t/x)$ for all $t \in D$ by FO2. It follows that $(M^*, s_T) \models \psi(t/x)$ for all $t \in D$ by induction hypothesis, and therefore $(M^*, s_T) \models (\forall x)\psi$. In the other direction, let $(M^*, s_T) \models (\forall x)\psi$ and assume the opposite, i.e., $\varphi = (\forall x)\psi \notin T$. Then, there exists some term $t \in D$ such that $(M^*, s_T) \models \neg\psi(t/x)$ (T is saturated), leading to a contradiction $(M^*, s_T) \not\models (\forall x)\psi$;
- Let $\varphi = P_{i,\geq r}\psi$. If $\varphi \in T$, then $\sup\{q \mid P_{i,\geq q}\psi \in T\} = \mu_{i,s_T}([\psi]) \geq r$ and $(M^*, s_T) \models P_{i,\geq r}\psi$. In the other direction, let $(M^*, s_T) \models P_{i,\geq r}\psi$, i.e., $\sup\{q \mid P_{i,\geq q}\psi \in T\} \geq r$. If $\mu_{i,s_T}([\psi]) > r$, then $P_{i,\geq r}\psi \in T$, because of the properties of supremum and monotonicity of the probability measure μ_{i,s_T}. If $\mu_{i,s_T}([\psi]) = r$, then $P_{i,\geq r}\psi \in T$ by Lemma 4.7(5).
- Suppose $\varphi = K_i\psi$. Let $K_i\psi \in T$. Since $\psi \in T/K_i$, then $\psi \in U$ for every U such that $s_T\mathcal{K}_i s_U$ (by the definition of $\mathcal{K}_i$). Therefore, $(M^*, s_U) \models \psi$ by induction hypothesis (ψ is subformula of $K_i\psi$), and then $(M^*, s_T) \models K_i\psi$.

  Let $(M^*, s_T) \models K_i\psi$. Assume the opposite, that $K_i\psi \notin T$. Then, $T/K_i \cup \{\neg\psi\}$ must be consistent. If it would not be consistent, then $T/K_i \vdash \psi$ by the Deduction theorem and $T \supset K_i(T/K_i) \vdash K_i\psi$ by Theorem 4.2, i.e., $K_i\psi \in T$, which is a contradiction. Therefore, $T/K_i \cup \{\neg\psi\}$ can be extended to a maximal consistent U, so $s_T\mathcal{K}_i s_U$. Since $\neg\psi \in U$, then $(M^*, s_U) \models \neg\psi$ by induction hypothesis, so we get the contradiction $(M^*, s_T) \not\models_{M^*} K_i\psi$.
- Observe that $\varphi = E_G\psi \in T$ iff $K_i\psi \in T$ for all $i \in G$ (by Lemma 4.8(1)) iff $(M^*, s_T) \models K_i\psi$ for all $i \in G$ (by previous case), i.e., $(M^*, s_T) \models E_G\psi$ (by the definition of $\models$ relation).
- $\varphi = C_G\psi \in T$ iff $(E_G)^m\psi \in T$ for all $m \in \mathbb{N}$ (by Lemma 4.8(3)) iff $(M^*, s_T) \models (E_G)^m\psi$ for all $m \in \mathbb{N}$ (by previous case), i.e., $(M^*, s_T) \models C_G\psi$.
- $\varphi = E^r_G\psi \in T$ iff $K^r_i\psi = K_i(P_{i,\geq r}\psi) \in T$ for all $i \in G$ (by Lemma 4.8(2)) iff $(M^*, s_T) \models K_i(P_{i,\geq r}\psi)$ (by the previous case $\varphi = K_i\psi$), i.e., $(M^*, s_T) \models E^r_G\psi$.
- Let $\varphi = (F^r_G)^m\psi$. Since $(F^r_G)^0\psi = \top$, the claim holds trivially. Also, $\varphi = (F^r_G)^{m+1}\psi = E^r_G(\psi \wedge (F^r_G)^m\psi) \in T$ iff $(M^*, s_T) \models E^r_G(\psi \wedge (F^r_G)^m\psi)$ (by the previous case), i.e., $(M^*, s_T) \models (F^r_G)^{m+1}\psi$, $m \in \mathbb{N}$.
- $\varphi = C^r_G\psi \in T$ iff $(F^r_G)^m\psi \in T$ for all $m \in \mathbb{N}$ (by Lemma 4.8(4)) iff $(M^*, s_T) \models (F^r_G)^m\psi$ for all $m \in \mathbb{N}$ (by the previous case), i.e., $(M^*, s_T) \models C_G\psi$. □

From Lemma 5.3 and Lemma 5.4, we immediately obtain the following corollary:

Theorem 5.5. $M^* \in \mathcal{M}^{MEAS}_{\mathcal{A}}$.

5.3 Completeness Theorem

Now, we state the main result of this article. In the following theorem, we summarize the results obtained above to prove the strong completeness of our axiomatic system for the class of measurable models.

Theorem 5.6 (Strong Completeness Theorem). *A theory T is consistent if and only if it is satisfiable in an $\mathcal{M}^{MEAS}_{\mathcal{A}}-$model.*

Proof. The direction from right to left is a consequence of the Soundness theorem. For the other direction, suppose that T is a consistent theory. We will show that T is satisfiable in the canonical model M^*, which belongs to $\mathcal{M}_{\mathcal{A}}^{MEAS}$, by Theorem 5.5. By Theorem 5.1, T can be extended to a saturated theory T^*. From Lemma 5.4, we have that $\varphi \in V$ iff $(M^*, s_V) \models \varphi$, for every saturated theory V. Consequently, $(M^*, s_{T^*}) \models \varphi$, for every $\varphi \in T^*$, and therefore $(M^*, s_{T^*}) \models T$. □

6 ADDING THE CONSISTENCY CONDITION

In the logic PCK^{fo} presented in this article, we proposed the most general case, where no relationship is posed between the modalities for knowledge and probability. Indeed, in the definition of the probability spaces $\mathcal{P}(i,s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ the sample space of possible events $S_{i,s}$ is an arbitrary nonempty subset of the set of all states S.

Now, we consider a natural additional assumption, called *consistency condition* in Reference [17], which forbids an agent to place a positive probability to the event she knows to be false. This assumption can be semantically captured by adding the condition $S_{i,s} \subseteq \mathcal{K}_i(s)$ to Definition 2.4. In the following definition, we introduce the corresponding subclass of measurable models $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$:

Definition 6.1. $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$ is the class of all measurable models $M = (S, D, I, \mathcal{K}, \mathcal{P}) \in \mathcal{M}_{\mathcal{A}}^{MEAS}$, such that

$$S_{i,s} \subseteq \mathcal{K}_i(s)$$

for all i and s, where $\mathcal{P}(i,s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$.

We will prove that adding the axiom[7]

CON. $K_i\varphi \rightarrow P_{i,\geq 1}\varphi$

to our axiomatization results in a system that is complete for the class of models $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$. Note that in that case, we can remove Probabilistic Necessitation from the list of inference rules, since, in presence of CON, it is derivable from Knowledge Necessitation. Indeed, the applications of the rules RK and RP are restricted to theorems only, so if $\vdash \varphi$, then $\vdash K_i\varphi$ by RK, and $\vdash P_{i,\geq 1}\varphi$ by CON. Thus, $\frac{\varphi}{P_{i,\geq 1}\varphi}$ is a derivable rule in the axiomatic system that we propose in the following definition:

Definition 6.2. The axiomatization $Ax_{PCK^{fo}}^{CON}$ consists of all the axiom schemata and inference rules from $Ax_{PCK^{fo}}$ except RP and, in addition, it contains the axiom CON.

The proposed axiomatic system is complete for the class of models $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$.

THEOREM 6.3. *The axiomatization $Ax_{PCK^{fo}}^{CON}$ is strongly complete for the class of models $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$.*

PROOF. The proof follows the idea of the proof of completeness of $Ax_{PCK^{fo}}$ for the class of models $\mathcal{M}_{\mathcal{A}}^{MEAS}$ presented above. Similarly as it is done in Section 5.1, we can show that any consistent theory T can be extended to a saturated theory in $Ax_{PCK^{fo}}^{CON}$ (not that the saturated theories in $Ax_{PCK^{fo}}^{CON}$ and $Ax_{PCK^{fo}}$ do not coincide; for example, the formula $K_i\varphi \wedge P_{i,<1}\varphi$ is consistent for the former axiomatization, but it is inconsistent for the later one). Then, we can construct the canonical model $M^* = (S, D, I, \mathcal{K}, \mathcal{P})$ using the saturated theories and prove the Truth lemma as in Section 5.2, and prove that $(M^*, s_{T^*}) \models T$ in the same way as in the proof of Theorem 5.6.

[7]This type of axiom is standard in logics in which probability is seen as an approximation of other modalities; for example, in probabilistic temporal logic, the axiom $G\varphi \rightarrow P_{\geq 1}\varphi$ ("if φ always holds, then its probability is equal to 1") is a part of axiomatization [37].

The problem is that M^* does not belong to the class $\mathcal{M}_{\mathcal{A}}^{MEAS,CON}$, since the condition $S_{i,s} \subseteq \mathcal{K}_i(s)$ is not ensured. Nevertheless, we can use M^* to obtain a model $M^{*'}$ from $M_{\mathcal{A}}^{MEAS,CON}$, in which T is also satisfied. We define $M^{*'}$ by modifying only the probability spaces $\mathcal{P}(i,s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ from M^* (i.e., S, D, I, and $\mathcal{K}$ are the same in both structures) in the following way: $M^{*'} = (S, D, I, \mathcal{K}, \mathcal{P}')$, such that

- $\mathcal{P}'(i,s) = (S'_{i,s}, \chi'_{i,s}, \mu'_{i,s})$, where
  $-S'_{i,s} = S \cap \mathcal{K}_i(s)$;
  $-\chi'_{i,s} = \{[\varphi]'_{i,s} \mid \varphi \in Sent_{PCK}\}$, where $[\varphi]' = [\varphi]_{i,s} \cap \mathcal{K}_i(s)$;
  $-$if $[\varphi]'_{i,s} \in \chi'_{i,s}$ then $\mu'_{i,s}([\varphi]'_{i,s}) = \mu_{i,s}([\varphi]_{i,s}) = \sup\{r \mid P_{i,\geq r}\varphi \in s\}$.

Now it only remains to prove that $M^{*'}$ is a model, i.e., that each $\mathcal{P}'(i,s)$ is a probability space, since the rest of the proof is trivial: $S_{i,s} \subseteq \mathcal{K}_i(s)$ obviously holds, and $(M^{*'}, s_{T^*}) \models T$ is ensured by the construction of $\mathcal{P}'$, and the fact that $(M^*, s_{T^*}) \models T$.

First, we show that every $\chi'_{i,s}$ is an algebra of sets using the corresponding results from the proof of Lemma 5.3(2)

- $[\varphi]'_{i,s} \cup [\psi]'_{i,s} = ([\varphi]_{i,s} \cap \mathcal{K}_i(s)) \cup ([\psi]_{i,s} \cap \mathcal{K}_i(s)) = ([\varphi]_{i,s} \cup [\psi]_{i,s}) \cap \mathcal{K}_i(s) = [\varphi \vee \psi]_{i,s} \cap \mathcal{K}_i(s) = [\varphi \vee \psi]'_{i,s} \in \chi'_{i,s}$;
- $S'_{i,s} \setminus [\varphi]'_{i,s} = S'_{i,s} \setminus ([\varphi]_{i,s} \cap \mathcal{K}_i(s))$, so from $S'_{i,s} = \mathcal{K}_i(s)$ and $[\varphi]_{i,s} = S \setminus [\neg\varphi]_{i,s}$, we obtain $S'_{i,s} \setminus [\varphi]'_{i,s} = [\neg\varphi]_{i,s} \cap \mathcal{K}_i(s) = [\neg\varphi]'_{i,s} \in \chi'_{i,s}$.

Finally, we prove that $\mu'_{i,s}$ is a finitely additive probability measure, for every i and s.

- $\mu'_{i,s}(S'_{i,s}) = ([\top]'_{i,s}) = \mu_{i,s}([\top]_{i,s}) = 1$;
- To prove finite additivity of $\mu'_{i,s}$, we need to prove that

$$\mu'_{i,s}([\varphi \vee \psi]'_{i,s}) = \mu'_{i,s}([\varphi]'_{i,s}) + \mu'_{i,s}([\psi]'_{i,s}) \tag{6.1}$$

  whenever

$$[\varphi]'_{i,s} \cap [\psi]'_{i,s} = \emptyset. \tag{6.2}$$

  The possible problem is that Equation (6.2) does not necessarily imply $[\varphi]_{i,s} \cap [\psi]_{i,s} = \emptyset$, so we cannot directly use finite additivity of $\mu_{i,s}$. However, we know that $\mu_{i,s}([\varphi \vee \psi]_{i,s}) = \mu_{i,s}([\varphi]_{i,s}) + \mu_{i,s}([\psi]_{i,s}) - \mu_{i,s}([\varphi \wedge \psi]_{i,s})$. Since $\mu'_{i,s}([\phi]'_{i,s}) = \mu_{i,s}([\phi]_{i,s})$ for every ϕ, to prove Equation (6.1) it is sufficient to show that

$$([\varphi \wedge \psi]_{i,s}) = 0. \tag{6.3}$$

  From Equation (6.2), we obtain $[\varphi]_{i,s} \cap [\psi]_{i,s} \cap \mathcal{K}_i(s) = \emptyset$, i.e., $[\varphi \wedge \psi]_{i,s} \cap \mathcal{K}_i(s) = \emptyset$. Consequently, $[\neg(\varphi \wedge \psi)]_{i,s} \subseteq \mathcal{K}_i(s)$, so $(M^*, t) \models \neg(\varphi \wedge \psi)$ for every $t \in \mathcal{K}_i(s)$, and $(M^*, s) \models K_i\neg(\varphi \wedge \psi)$. Since s is a saturated theory, from the Truth lemma, we have $K_i\neg(\varphi \wedge \psi) \in s$, and consequently $P_{i,\geq 1}\neg(\varphi \wedge \psi) \in s$, by CON. Then, $\mu_{i,s}([\varphi]_{i,s}) = \sup\{r \mid P_{i,\geq r}\varphi \in s\} = 1$, which implies Equation (6.3). □

Remark 5. Apart from consistency condition, Fagin and Halpern [17] consider other relations between the sample space $S_{i,s}$ and possible worlds $K_i(s)$, which model some typical situations in the multi-agent systems. They also provide their characterization by the corresponding axioms.

First they analyze the situations in which the probabilities of the events are common knowledge, i.e, there is a unique, collective, and objective view on the probability of the events. Then the agents in the same state share the same known probability spaces, which is captured by the condition of *objectivity*: $\mathcal{P}(i,s) = \mathcal{P}(j,s)$ for all i, j, and s.

Second, they model the situation where an agent uses the same probability space in all the worlds he considers possible. This situation occurs when no nonprobabilistic choices are made to cause different probability distributions in the possible worlds. The corresponding condition, called *state determined property*, says that if $t \in \mathcal{K}_i(s)$, then $\mathcal{P}(i,s) = \mathcal{P}(i,t)$.

Third, sometimes the nonprobabilistic choices happen and induce varied probability spaces. Then the possible worlds could be divided into partitions that share the same probability distributions, after such choice has been made. This case is specified by the condition of *uniformity*: if $\mathcal{P}(i,s) = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ and $t \in \mathcal{S}_{i,s}$, then $\mathcal{P}(i,s) = \mathcal{P}(i,t)$.

Similarly as we have done with the consistency condition, we can also characterize the three above mentioned conditions by adding corresponding axioms to our axiomatic system. It is straightforward to check that the following axioms, which are similar to the ones proposed in Reference [17], capture the mentioned relations between modalities of knowledge and probability:

$P_{i,\geq r}\varphi \rightarrow P_{j,\geq r}\varphi$ (objectivity),
$P_{i,\geq r}\varphi \rightarrow K_i P_{i,\geq r}\varphi$ (state determined property),
$P_{i,\geq r}\varphi \rightarrow P_{i,\geq 1} P_{i,\geq r}\varphi$ (uniformity).

7 CONCLUSION

The starting points for our research were References [17, 26], where weakly complete axiomatizations for a propositional logic combining knowledge and probability, and a non-probabilistic propositional logic for knowledge with infinitely many agents (respectively), are presented. We combine those two approaches and extend both of them to the logic PCK^{fo} with an expressive first-order language.

We provide a sound and strongly complete axiomatization $Ax_{PCK^{fo}}$ for the corresponding semantics of PCK^{fo}. Since any reasonable, semantically defined first-order epistemic logic with common knowledge is not recursively axiomatizable [49], we propose the axiomatization with infinitary rules of inference, and we obtain completeness modifying the standard Henkin construction of saturated extensions of consistent theories. In the logic PCK^{fo}, we consider the most general semantics, with independent modalities for knowledge and probability. We also show how to extend the set of axioms and modify the axiomatization technique to capture models in which agents assign probabilities only to the sets of worlds they consider possible. We also give hints how to extend our axiomatization in several different ways to capture other interesting relationships between the modalities for knowledge and probability, considered in Reference [17].

In this article, we use the semantic definition of the probabilistic common knowledge operator C_G^r proposed by Fagin and Halpern [17]. As we have mentioned in Section 2.2, Monderer and Samet [35] proposed a different definition, where probabilistic common knowledge is equivalent to the infinite conjunction of the formulas $E_G^r\varphi, (E_G^r)^2\varphi, (E_G^r)^3\varphi \ldots$ It is easy to check that our axiomatization $Ax_{PCK^{fo}}$ can be easily modified to capture the definition of Monderer and Samet. Namely, the axiom APC and rule RPC should be replaced with the axiom $C_G^r\varphi \rightarrow (E_G^r)^m\varphi$, $m \in \mathbb{N}$ and the inference rule $\frac{\{\Phi_{k,\theta,\mathrm{X}}((E_G^r)^m\varphi) \mid m \in \mathbb{N}\}}{\Phi_{k,\theta,\mathrm{X}}(C_G^r\varphi)}$.

Finally, although this article is focused on the issue of providing a strongly complete axiomatization, we should also mention that logics of this type could be used to reason about various distributed systems with interacting agents [20]. More recently, Reference [32] proposes a temporal epistemic logic with a non-rigid set of agents for analyzing the blockchain protocol, while for the same protocol Reference [25] uses common knowledge about probabilities to describe conditions for achieving consensus on a public ledger. It may happen that probabilistic common knowledge can provide additional insight into the convergence for reaching the consensus in that framework.

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